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AND

DISCUSSIONS

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Current Papers and Discussions

	Published	Discussion closes
Symposium: Dewatering, Incineration, and Use of Sewage Sludge	Jan., 1943	
Discussion in Feb., June, Sept., 1943, Feb. 1944		Closed*
Symposium: Rainbow Arch Bridge Over Niagara Gorge	Oct., 1943	
Discussion in Dec., 1943, Jan., Feb., Mar., Apr., May, June, 1944		Closed*
Sergev, S. Shear Effect on the Strength of Struts	Nov., 1943	
Discussion in Feb., Mar., Oct., 1944		Closed
Elder, Clay C. Original and Readjusted Repayment Contracts, Boulder Canyon Project	Dec., 1943	
Discussion in Jan., Feb., 1944		Closed*
Edwards, F. W., and Soucek, Edward. Surges in Panama Canal Reproduced in Model	Jan., 1944	
Discussion in Apr., May, June, 1944		Closed*
Huntting, Marshall T. Geology in Highway Engineering	Dec., 1943	
Discussion in Feb., Mar., Apr., May, June, Sept., 1944		Closed*
Eisenlohr, William S., Jr. Coefficients for Velocity Distribution in Open-Channel Flow	Jan., 1944	
Discussion in Apr., Sept., 1944		Closed*
Symposium: Military Airfields	Jan., 1944	
Discussion in Feb., Mar., Apr., May, June, Sept., 1944		Closed*
Report: Principles of Depreciation: Report of the Special Committee Authorized by the Board of Direction to Analyze and Discuss the 1943 Report of the National Association of Railroad and Utilities Commissioners' Committee on Depreciation	June, 1944	
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Note.—The closing dates herein published are final except when names of prospective discussers are registered for special extention of time.

* Publication of closing discussion pending.

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PAPERS

DISCUSSIONS

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Those who are planning papers for submission to PROCEEDINGS *will expedite Committee action measurably by first studying the standard instructions as to style, content, and format. For reprints, address the Secretary, Committee on Publications, 33 W. 39th St., New York 18, N. Y.*

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

DESIGN CONSTANTS FOR BEAMS WITH NONSYMMETRICAL STRAIGHT HAUNCHES

By August L. Ahlf,[1] Jun. Am. Soc. C. E.

Synopsis

Tables are presented in this paper, with outlines of the procedure to be followed in their use, to assist the structural designer in determining the stiffness, carry-over factors, and fixed-end moments for various loadings of beams having straight haunches placed nonsymmetrically with respect to the center of the span. Although this work was performed to determine these factors for nonsymmetrically haunched beams, it is also applicable to beams having straight haunches at only one end and to symmetrical cases as well.

The Method

Hardy Cross,[2] M. Am. Soc. C. E., and others[3] have shown that the column analogy method may be used to obtain the stiffness and carry-over factors and also the fixed-end moments for structural members having either constant or variable cross sections throughout their length.

The properties that must be obtained for the analogous column are the area, the location of the centroidal axis, and the second moment of the area about the centroidal axis. When these have been determined, the stiffness and carry-over factors may be obtained.

To determine the fixed-end moments, it is also necessary to obtain the area under the $\frac{M}{E\,I}$-curve corresponding to the loading and the dimensions of the beam as well as the location of the centroidal axis of the $\frac{M}{E\,I}$-diagram.

When the beam under consideration has haunches at the ends, which haunches are not symmetrical with respect to the center of the span, the

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March 1, 1945.**

[1] Engr. U. S. Bureau of Reclamation, Dept. of Interior, Denver, Colo.

[2] "Continuous Frames of Reinforced Concrete," by Hardy Cross and N. D. Morgan, John Wiley & Sons, Inc., New York, N. Y., 1932.

[3] "Theory of Modern Steel Structures," by L. E. Grinter, Vol. 2, Macmillan Co., New York, N. Y., 1937.

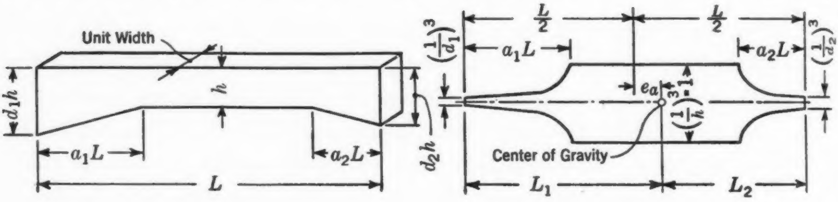
computations required to obtain the properties of the analogous column and its loading diagram may become laborious and time consuming, particularly if accurate results are desired.

The tabular values herein presented have been prepared to facilitate the computations. By their use, the properties of the analogous column and the $\frac{M}{E\,I}$-diagram may be obtained by simple arithmetical combinations.

Notation.—The letter symbols adopted for use in this paper are defined where they first appear, either in the text or by diagrams, and are assembled in an Appendix for convenience of reference.

The Analogous Column

A beam, having straight haunches placed nonsymmetrically with respect to the center of the span, and the corresponding analogous column are shown in Fig. 1. The properties of the analogous column were obtained by integrating



(*a*) BEAM WITH NON-SYMMETRICAL STRAIGHT HAUNCHES (*b*) ANAGOLOUS COLUMN

Fig. 1.

the $\frac{1}{E\,I}$-diagram, which was divided into the four parts shown in Fig. 2. The properties of each segment were determined separately and then combined to obtain the desired properties at each end. The total area of the analogous column was found to be

$$A = \frac{L}{E\,I_o}(C_1 + C_2) \quad\quad (1)$$

in which I_o is the moment of inertia of the central section of a beam. The dimensionless constants, C_1 and C_2, for this case, are the coefficients pertaining to ends 1 and 2, respectively. These constants (see Table 1(*a*)) are expressed by

$$C = \left(\frac{d+1}{2\,d^2} - 1\right)a + \frac{1}{2} \quad\quad (2)$$

Eq. 2 is made applicable to either end 1 or 2 by inserting the appropriate subscript on each symbol. This is true also of subsequent equations for C. Table 1(*a*) was prepared to yield values of C for various values of a and d, applying the appropriate subscript.

The total statical moment of the area of the analogous column about the center of the span was found to be

$$Q_I = \frac{L^2}{E\,I_o}(-C_1 + C_2) \quad\quad (3)$$

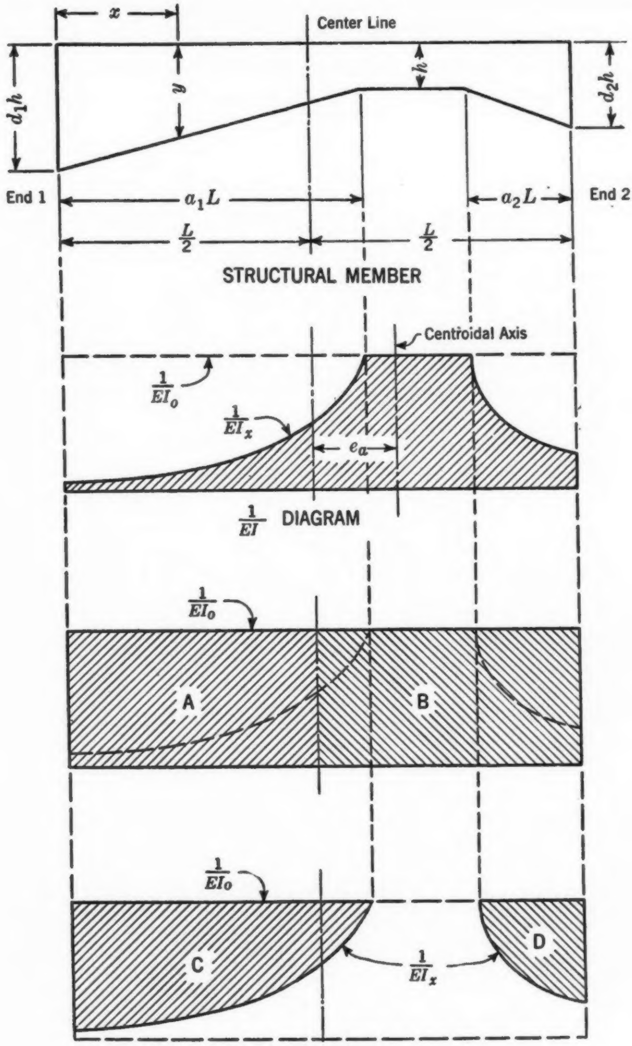


Fig. 2.

in which the factor C (with appropriate subscript) is expressed by:

$$C = + \left(-\frac{1}{2} + \frac{1}{4\,d} + \frac{1}{4\,d^2} \right) a - \left(\frac{1}{2\,d} - \frac{1}{2} \right) a^2 + \frac{1}{8} \quad \text{(4)}$$

Table 1(b) was prepared to yield values of C for various values of a and d in Eq. 4. The total second moment of the area of the analogous column about the center of the span is:

$$Q_{II} = \frac{L^3}{E\,I_o} (C_1 + C_2) \quad \text{(5)}$$

TABLE 1.—VALUES OF THE DIMENSIONLESS CONSTANT C FOR BEAMS WITH STRAIGHT HAUNCHES

(Apply Subscripts 1 and 2 to Designate Ends 1 and 2, Respectively, of a Beam)

d	VALUES OF a:										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
(*a*) AREA OF THE ANALOGOUS COLUMN (EQ. 2)											
1.0	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
1.2	0.5000	0.4764	0.4528	0.4292	0.4056	0.3819	0.3583	0.3347	0.3111	0.2875	0.2639
1.4	0.5000	0.4612	0.4224	0.3837	0.3449	0.3061	0.2673	0.2286	0.1898	0.1510	0.1122
1.6	0.5000	0.4508	0.4016	0.3523	0.3031	0.2539	0.2047	0.1555	0.1063	0.0570	0.0078
1.8	0.5000	0.4432	0.3884	0.3296	0.2728	0.2160	0.1593	0.1025	0.0456	−0.0111	−0.0679
2.0	0.5000	0.4375	0.3750	0.3125	0.2500	0.1875	0.1250	0.0625	0.0000	−0.0625	−0.1250
2.2	0.5000	0.4331	0.3661	0.2992	0.2322	0.1653	0.0983	0.0314	−0.0355	−0.1025	−0.1694
2.4	0.5000	0.4295	0.3590	0.2885	0.2181	0.1476	0.0771	0.0066	−0.0639	−0.1344	−0.2049
2.6	0.5000	0.4266	0.3533	0.2799	0.2065	0.1331	0.0598	−0.0136	−0.0870	−0.1604	−0.2337
2.8	0.5000	0.4242	0.3485	0.2727	0.1969	0.1212	0.0454	−0.0304	−0.1081	−0.1819	−0.2576
3.0	0.5000	0.4222	0.3444	0.2667	0.1889	0.1111	0.0333	−0.0444	−0.1222	−0.2000	−0.2778
4.0	0.5000	0.4156	0.3313	0.2469	0.1625	0.0781	−0.0063	−0.0906	−0.1750	−0.2594	−0.3438
(*b*) FIRST MOMENT OF THE ANALOGOUS COLUMN (EQ. 4)											
1.0	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250
1.2	0.1250	0.1142	0.1047	0.0971	0.0911	0.0868	0.0842	0.0832	0.0839	0.0862	0.0902
1.4	0.1250	0.1070	0.0919	0.0797	0.0703	0.0638	0.0601	0.0593	0.0613	0.0662	0.0740
1.6	0.1250	0.1023	0.0833	0.0681	0.0566	0.0489	0.0448	0.0446	0.0481	0.0554	0.0664
1.8	0.1250	0.0988	0.0771	0.0598	0.0470	0.0386	0.0346	0.0351	0.0400	0.0494	0.0630
2.0	0.1250	0.0962	0.0725	0.0537	0.0400	0.0312	0.0265	0.0287	0.0350	0.0462	0.0625
2.2	0.1250	0.0942	0.0690	0.0491	0.0347	0.0258	0.0224	0.0243	0.0318	0.0447	0.0631
2.4	0.1250	0.0927	0.0662	0.0455	0.0307	0.0217	0.0185	0.0212	0.0298	0.0440	0.0643
2.6	0.1250	0.0914	0.0639	0.0426	0.0275	0.0185	0.0157	0.0190	0.0284	0.0440	0.0657
2.8	0.1250	0.0903	0.0621	0.0403	0.0249	0.0160	0.0134	0.0173	0.0276	0.0445	0.0677
3.0	0.1250	0.0894	0.0605	0.0383	0.0227	0.0139	0.0117	0.0161	0.0272	0.0450	0.0694
4.0	0.1250	0.0866	0.0556	0.0322	0.0162	0.0079	0.0069	0.0135	0.0275	0.0498	0.0781
(*c*) SECOND MOMENT OF THE ANALOGOUS COLUMN (EQ. 6)											
1.0	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417
1.2	0.0417	0.0365	0.0328	0.0303	0.0286	0.0275	0.0269	0.0263	0.0256	0.0244	0.0226
1.4	0.0417	0.0333	0.0274	0.0234	0.0209	0.0194	0.0185	0.0177	0.0166	0.0147	0.0116
1.6	0.0417	0.0312	0.0238	0.0190	0.0160	0.0143	0.0133	0.0125	0.0111	0.0086	0.0043
1.8	0.0417	0.0296	0.0212	0.0158	0.0126	0.0110	0.0100	0.0091	0.0075	0.0045	−0.0008
2.0	0.0417	0.0285	0.0194	0.0135	0.0102	0.0086	0.0077	0.0068	0.0049	0.0014	−0.0047
2.2	0.0417	0.0275	0.0179	0.0118	0.0084	0.0068	0.0060	0.0050	0.0031	−0.0010	−0.0078
2.4	0.0417	0.0268	0.0169	0.0105	0.0072	0.0056	0.0049	0.0039	0.0017	−0.0027	−0.0101
2.6	0.0417	0.0263	0.0159	0.0096	0.0061	0.0046	0.0040	0.0029	0.0006	−0.0042	−0.0123
2.8	0.0417	0.0258	0.0152	0.0087	0.0053	0.0039	0.0033	0.0023	−0.0002	−0.0052	−0.0138
3.0	0.0417	0.0254	0.0145	0.0081	0.0046	0.0033	0.0027	0.0016	−0.0009	−0.0063	−0.0155
4.0	0.0417	0.0242	0.0127	0.0061	0.0030	0.0017	0.0013	0.0003	−0.0028	−0.0092	−0.0201
(*d*) LOAD CONSTANTS FOR UNIFORM LOADS (EQ. 19)											
1.0	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417	0.0417
1.2	0.0417	0.0413	0.0402	0.0385	0.0364	0.0339	0.0313	0.0286	0.0260	0.0232	0.0216
1.4	0.0417	0.0410	0.0391	0.0363	0.0327	0.0286	0.0242	0.0197	0.0154	0.0115	0.0082
1.6	0.0417	0.0408	0.0383	0.0346	0.0299	0.0246	0.0189	0.0132	0.0077	0.0028	−0.0012
1.8	0.0417	0.0406	0.0377	0.0333	0.0278	0.0216	0.0149	0.0083	0.0020	−0.0036	−0.0080
2.0	0.0417	0.0405	0.0372	0.0323	0.0261	0.0192	0.0118	0.0045	−0.0025	−0.0085	−0.0132
2.2	0.0417	0.0404	0.0368	0.0315	0.0249	0.0173	0.0093	0.0014	−0.0060	−0.0133	−0.0173
2.4	0.0417	0.0403	0.0366	0.0309	0.0238	0.0157	0.0073	−0.0010	−0.0088	−0.0154	−0.0204
2.6	0.0417	0.0403	0.0362	0.0303	0.0228	0.0143	0.0055	−0.0032	−0.0112	−0.0180	−0.0231
2.8	0.0417	0.0402	0.0361	0.0297	0.0220	0.0132	0.0041	−0.0048	−0.0131	−0.0201	−0.0252
3.0	0.0417	0.0401	0.0358	0.0293	0.0214	0.0123	0.0029	−0.0064	−0.0148	−0.0219	−0.0270
4.0	0.0417	0.0399	0.0351	0.0279	0.0189	0.0090	−0.0013	−0.0114	−0.0207	−0.0277	−0.0326

TABLE 1.—(*Continued*)

(e) Statical Moment Constants for Uniform Load (Eq. 20)

d	Values of a: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
1.2	0.0078	0.0076	0.0072	0.0066	0.0062	0.0058	0.0056	0.0054	0.0056	0.0058	0.0061
1.4	0.0078	0.0075	0.0068	0.0058	0.0051	0.0045	0.0042	0.0040	0.0044	0.0048	0.0053
1.6	0.0078	0.0074	0.0065	0.0053	0.0044	0.0036	0.0032	0.0032	0.0037	0.0043	0.0050
1.8	0.0078	0.0073	0.0063	0.0049	0.0038	0.0030	0.0026	0.0026	0.0033	0.0041	0.0049
2.0	0.0078	0.0073	0.0061	0.0046	0.0034	0.0025	0.0021	0.0022	0.0030	0.0039	0.0049
2.2	0.0078	0.0072	0.0060	0.0043	0.0030	0.0021	0.0018	0.0019	0.0028	0.0039	0.0050
2.4	0.0078	0.0072	0.0058	0.0041	0.0027	0.0018	0.0015	0.0017	0.0028	0.0040	0.0051
2.6	0.0078	0.0072	0.0057	0.0039	0.0025	0.0016	0.0013	0.0016	0.0027	0.0040	0.0052
2.8	0.0078	0.0071	0.0056	0.0038	0.0024	0.0014	0.0011	0.0015	0.0027	0.0041	0.0053
3.0	0.0078	0.0071	0.0056	0.0036	0.0022	0.0012	0.0010	0.0014	0.0027	0.0042	0.0054
4.0	0.0078	0.0070	0.0053	0.0032	0.0017	0.0007	0.0006	0.0013	0.0029	0.0046	0.0060

(f) Load Constants for Triangular Load with Small End Adjacent to Haunch (Eq. 21a)

d	Values of a: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	0.0365	0.0365	0.0365	0.0365	0.0365	0.0365	0.0365	0.0365	0.0365	0.0365	0.0365
1.2	0.0365	0.0362	0.0351	0.0340	0.0322	0.0301	0.0275	0.0249	0.0225	0.0195	0.0172
1.4	0.0365	0.0360	0.0337	0.0323	0.0290	0.0251	0.0220	0.0172	0.0132	0.0087	0.0052
1.6	0.0365	0.0358	0.0332	0.0310	0.0266	0.0214	0.0164	0.0110	0.0060	0.0010	−0.0044
1.8	0.0365	0.0357	0.0326	0.0300	0.0250	0.0190	0.0130	0.0068	0.0010	−0.0055	−0.0111
2.0	0.0365	0.0356	0.0321	0.0292	0.0235	0.0172	0.0102	0.0032	−0.0035	−0.0111	−0.0165
2.2	0.0365	0.0355	0.0319	0.0287	0.0225	0.0158	0.0080	0.0002	−0.0080	−0.0155	−0.0207
2.4	0.0365	0.0355	0.0317	0.0280	0.0216	0.0144	0.0058	−0.0022	−0.0110	−0.0190	−0.0241
2.6	0.0365	0.0354	0.0315	0.0275	0.0209	0.0133	0.0042	−0.0044	−0.0132	−0.0219	−0.0269
2.8	0.0365	0.0354	0.0314	0.0270	0.0202	0.0123	0.0030	−0.0060	−0.0150	−0.0234	−0.0290
3.0	0.0365	0.0353	0.0312	0.0268	0.0196	0.0115	0.0018	−0.0074	−0.0165	−0.0247	−0.0306
4.0	0.0365	0.0352	0.0304	0.0256	0.0175	0.0084	−0.0020	−0.0123	−0.0220	−0.0307	−0.0366

(g) Statical Moment Constants for Triangular Load with Small End Adjacent to Haunch (Eq. 22a)

d	Values of a: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064
1.2	0.0064	0.0063	0.0060	0.0056	0.0052	0.0047	0.0046	0.0047	0.0048	0.0049	0.0051
1.4	0.0064	0.0062	0.0056	0.0050	0.0044	0.0038	0.0034	0.0036	0.0039	0.0042	0.0048
1.6	0.0064	0.0061	0.0053	0.0046	0.0038	0.0031	0.0025	0.0028	0.0033	0.0039	0.0048
1.8	0.0064	0.0061	0.0050	0.0043	0.0034	0.0026	0.0020	0.0023	0.0031	0.0039	0.0049
2.0	0.0064	0.0060	0.0048	0.0040	0.0031	0.0022	0.0016	0.0020	0.0030	0.0040	0.0052
2.2	0.0064	0.0060	0.0047	0.0038	0.0028	0.0018	0.0013	0.0017	0.0030	0.0042	0.0055
2.4	0.0064	0.0060	0.0045	0.0036	0.0026	0.0016	0.0011	0.0016	0.0030	0.0044	0.0057
2.6	0.0064	0.0059	0.0044	0.0035	0.0024	0.0014	0.0010	0.0015	0.0030	0.0045	0.0058
2.8	0.0064	0.0059	0.0043	0.0034	0.0023	0.0012	0.0009	0.0015	0.0030	0.0046	0.0060
3.0	0.0064	0.0059	0.0043	0.0033	0.0021	0.0011	0.0008	0.0015	0.0030	0.0047	0.0062
4.0	0.0064	0.0059	0.0041	0.0030	0.0018	0.0007	0.0004	0.0013	0.0030	0.0049	0.0069

(h) Load Constants for Triangular Load with Large End Adjacent to Haunch (Eq. 21b)

d	Values of a: 0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	0.0469	0.0469	0.0469	0.0469	0.0469	0.0469	0.0469	0.0469	0.0469	0.0469	0.0469
1.2	0.0469	0.0464	0.0453	0.0430	0.0406	0.0377	0.0351	0.0323	0.0295	0.0269	0.0260
1.4	0.0469	0.0460	0.0445	0.0403	0.0364	0.0321	0.0264	0.0222	0.0176	0.0143	0.0112
1.6	0.0469	0.0458	0.0434	0.0382	0.0332	0.0278	0.0214	0.0154	0.0094	0.0046	0.0020
1.8	0.0469	0.0455	0.0428	0.0366	0.0306	0.0242	0.0168	0.0098	0.0030	−0.0017	−0.0049
2.0	0.0469	0.0454	0.0423	0.0354	0.0287	0.0212	0.0134	0.0058	−0.0015	−0.0059	−0.0099
2.2	0.0469	0.0453	0.0417	0.0343	0.0273	0.0188	0.0106	0.0026	−0.0040	−0.0090	−0.0139
2.4	0.0469	0.0452	0.0415	0.0338	0.0260	0.0170	0.0088	0.0002	−0.0066	−0.0118	−0.0167
2.6	0.0469	0.0451	0.0409	0.0331	0.0247	0.0153	0.0068	−0.0020	−0.0092	−0.0141	−0.0193
2.8	0.0469	0.0450	0.0408	0.0324	0.0238	0.0141	0.0052	−0.0036	−0.0112	−0.0168	−0.0214
3.0	0.0469	0.0449	0.0404	0.0318	0.0232	0.0131	0.0040	−0.0054	−0.0131	−0.0191	−0.0234
4.0	0.0469	0.0446	0.0398	0.0302	0.0203	0.0096	−0.0016	−0.0105	−0.0194	−0.0247	−0.0286

TABLE 1.—(*Continued*)

d	Values of a:										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
(*i*) Statical Moment Constants for Triangular Load with Large End Adjacent to Haunch (Eq. 22*b*)											
1.0	0.0092	0.0092	0.0092	0.0092	0.0092	0.0092	0.0092	0.0092	0.0092	0.0092	0.0092
1.2	0.0092	0.0090	0.0084	0.0079	0.0072	0.0068	0.0066	0.0065	0.0066	0.0068	0.0070
1.4	0.0092	0.0088	0.0079	0.0070	0.0060	0.0053	0.0050	0.0049	0.0052	0.0055	0.0058
1.6	0.0092	0.0087	0.0075	0.0063	0.0052	0.0042	0.0039	0.0038	0.0041	0.0047	0.0052
1.8	0.0092	0.0086	0.0071	0.0058	0.0045	0.0034	0.0031	0.0029	0.0035	0.0042	0.0048
2.0	0.0092	0.0085	0.0069	0.0053	0.0040	0.0028	0.0025	0.0025	0.0031	0.0039	0.0046
2.2	0.0092	0.0084	0.0067	0.0050	0.0036	0.0024	0.0022	0.0021	0.0028	0.0038	0.0045
2.4	0.0092	0.0084	0.0066	0.0047	0.0033	0.0021	0.0019	0.0018	0.0026	0.0036	0.0045
2.6	0.0092	0.0083	0.0065	0.0045	0.0031	0.0019	0.0017	0.0017	0.0025	0.0035	0.0045
2.8	0.0092	0.0083	0.0064	0.0044	0.0029	0.0017	0.0015	0.0016	0.0024	0.0034	0.0045
3.0	0.0092	0.0083	0.0063	0.0042	0.0028	0.0015	0.0013	0.0015	0.0023	0.0033	0.0046
4.0	0.0092	0.0082	0.0060	0.0037	0.0022	0.0007	0.0003	0.0010	0.0021	0.0032	0.0051

in which the factor C (with appropriate subscript) is expressed by:

$$C = \frac{1}{24} + \left[\left(\frac{1}{d - 1} \right)^3 \left(\log_e d + \frac{2}{d} - \frac{1}{2\,d^2} - \frac{3}{2} \right) + \left(\frac{d - 1}{2\,d^2} - \frac{1}{3} \right) \right] a^3 + \left(\frac{1}{2} - \frac{1}{2\,d} \right) a^2 + \left(\frac{d + 1}{8\,d^2} - \frac{1}{4} \right) a \qquad (6)$$

Table 1(*c*) was prepared to yield values of C for various values of a and d in Eq. 6.

The location of the centroidal axis of the total area of the analogous column from the center of the span may be obtained by the relation,

$$e_a = \frac{Q_{\mathrm{I}}}{A} \qquad (7)$$

The eccentric distance e_a will be positive if the centroidal axis is toward end 2 from the center of the span and negative if it is toward end 1. The longer distance from the centroidal axis to the end of a beam (end 1, Fig. 2) is expressed by:

$$L_1 = -\frac{L}{2} - e_a \qquad (8a)$$

Similarly the remaining distance from the centroidal axis to the end of a beam (see end 2, Fig. 2) is

$$L_2 = \frac{L}{2} - e_a \qquad (8b)$$

If values of L_1 are always considered to be negative and values of L_2 are always considered to be positive, the signs in the column formula will follow automatically.

The total second moment Q_a of the area of the analogous column, taken about the centroidal axis, is

$$Q_a = Q_{\mathrm{II}} - e_a\,Q_{\mathrm{I}} \qquad (9)$$

Stiffness Factors

The "absolute stiffness factor" is defined herein as that bending moment required to rotate the free end of a beam through an angle of one radian while the opposite end remains fixed. It may be shown that the absolute stiffness factor for a member having a uniform cross section throughout its length is equal to $\frac{4\,E\,I}{L}$. However, if the member is not prismatic, but has a variable moment of inertia along its length, the coefficient of the absolute stiffness will be something other than 4. This factor will be different for the two ends unless the haunches at the two ends have the same dimensions.

When the modulus of elasticity, E, of the several members of the frame is the same and each of the members is a prismatic one, the quantity $4\,E$ will be constant and, therefore, can be eliminated from the computations. For this reason it has become common practice to regard the stiffness of a member as being equal to I/L. Herein this factor is called the relative stiffness factor. The relative stiffness factor may be obtained by dividing the absolute stiffness factor by $4\,E$.

When using the analogous column, the rotation of the structural member must be represented as a reaction or load and the bending moments are represented as fiber stresses. The stiffness factor at either end may be found by applying a unit load at that end and computing the extreme fiber stress at the corresponding end by the column formula,

$$f = \frac{P}{A} \pm \frac{P\,e\,C}{Q_a} \qquad \text{(10)}$$

The load applied in this instance represents the unit load, P, of one radian; the fiber stress, f, represents the absolute stiffness factor; and the distance, C, is the distance from the center of gravity of the analogous column to the end of the beam where the stiffness factor is desired. The absolute stiffness at end 1 is, then:

$$K_1 = \frac{1}{A} + \frac{L^2_1}{Q_a} \qquad \text{(11a)}$$

The relative stiffness at end 1 is

$$k_1 = \frac{K_1}{4\,E} \qquad \text{(11b)}$$

Similarly, the absolute stiffness at end 2 is

$$K_2 = \frac{1}{A} + \frac{L^2_2}{Q_a} \qquad \text{(12a)}$$

and the relative stiffness at end 2 is

$$k_2 = \frac{K_2}{4\,E} \qquad \text{(12b)}$$

Carry-Over Factors

If the end of a member, which is upon unyielding supports at both ends, is rotated while the other end is held fixed, a moment will be induced into the

fixed end. The induced moment is defined as the "carry-over moment," and that proportion of the applied moment which is carried over is defined as the "carry-over factor."

The "carry-over moment" induced at the far end of the member may also be computed by the column formula:

$$N = \frac{1}{A} + \frac{L_1 L_2}{Q_a} \qquad (13)$$

The carry-over factor at end 1 may be obtained from:

$$r_1 = \frac{N}{K_1} \qquad (14a)$$

and the carry-over factor at end 2 may be obtained from:

$$r_2 = \frac{N}{K_2} \qquad (14b)$$

Fixed-End Moments

The magnitude and point of application of the load placed on the analogous column to determine the fixed-end moment were obtained by integrating the $\frac{M}{E\,I}$-diagram shown in Fig. 3. The diagram was divided into the four parts shown. The load and statical moment about the center of the span were obtained for each part separately, and then combined to obtain the load and statical moment of the two ends. The load representing the total $\frac{M}{E\,I}$-diagram was found to be

$$P = \frac{W\,L^2}{E\,I_o}\,(C_1 + C_2) \qquad (15)$$

The equations representing the values of these constants for the various types of loading are given subsequently under the discussion of the particular type of loading. Tables 1(*d*), 1(*f*), and 1(*h*) contain values of C for various values of a and d, and for several of the more common loadings, in the application of Eq. 15.

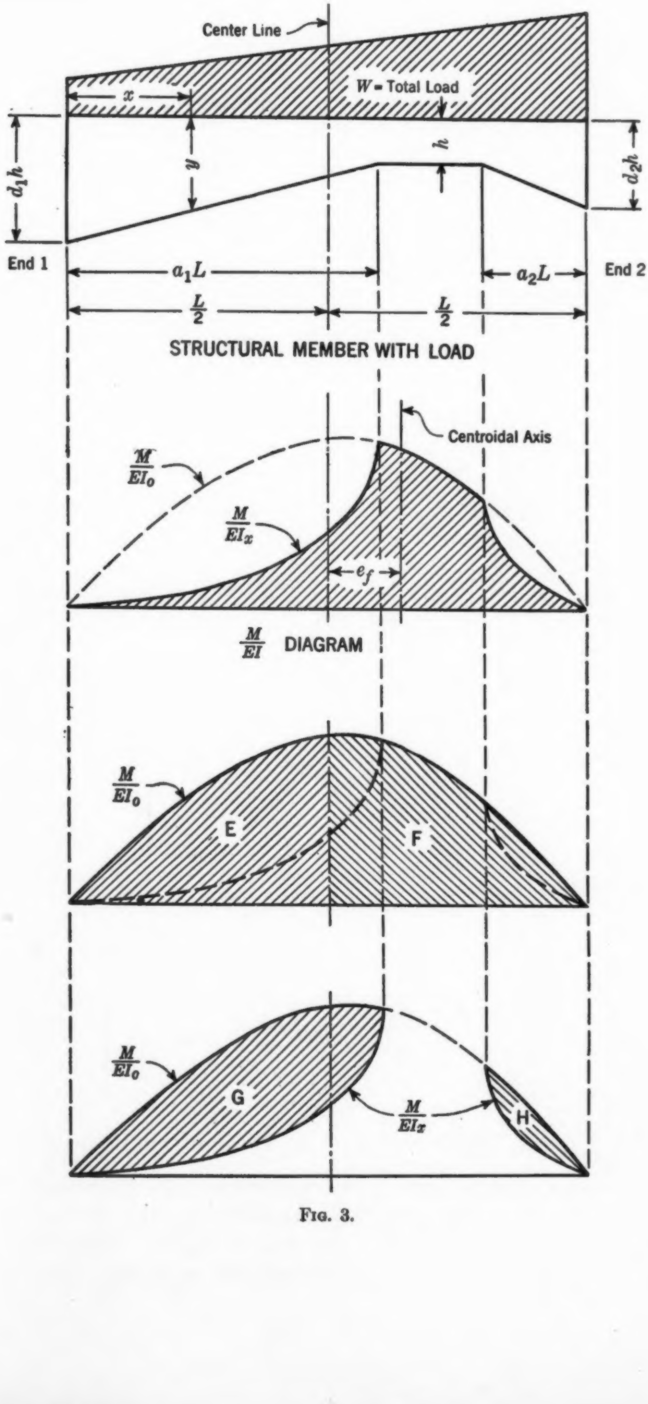
The following was found to represent the statical moment of the total $\frac{M}{E\,I}$-diagram about the center of the span:

$$Q_f = \frac{W\,L^3}{E\,I_o}\,(-\,C_1 + C_2) \qquad (16)$$

The equations representing the values of these constants for the various types of loading are given subsequently under the discussion of the particular type of loading.

Values of C in Eq. 16, for various values of a and d for several of the more common types of loading, are given in Tables 1(*e*), 1(*g*), and 1(*i*).

The location of the centroidal axis of the total $\frac{M}{E\,I}$-diagram from the center



Center Line
W = Total Load
x
y
h
d_1h
d_2h
End 1
End 2
a_1L
a_2L
$\frac{L}{2}$
$\frac{L}{2}$
STRUCTURAL MEMBER WITH LOAD
Centroidal Axis
$\frac{M}{EI_0}$
$\frac{M}{EI_x}$
e_f
$\frac{M}{EI}$ DIAGRAM
$\frac{M}{EI_0}$
E
F
$\frac{M}{EI_0}$
G
$\frac{M}{EI_x}$
H

FIG. 3.

of the span may be obtained by

$$e_f = \frac{Q_f}{P} \qquad (17)$$

The value of e_f will be positive if the centroidal axis is toward end 2 from the center of the span and negative if it is toward end 1.

The fixed-end moment at end 1 may be computed from the column formula:

$$\text{FEM} = \frac{P}{A} + \frac{(e_f - e_a)\, P\, L_1}{Q_a} \qquad (18a)$$

Similarly, the fixed-end moment at end 2 may be computed by:

$$\text{FEM} = \frac{P}{A} + \frac{(e_f - e_a)\, P\, L_2}{Q_a} \qquad (18b)$$

Fixed-End Moment with Uniform Load

The value of the constant C necessary for determining the area of the $\frac{M}{E\,I}$-diagram for the case of a uniform load (Table 1(d)) was found to be

$$C = \frac{1}{24} + \left[\frac{\frac{d}{2} - 1 + \frac{1}{2\,d}}{2\,(1-d)^2} - \frac{1}{4}\right] a^2 + \left[\frac{1}{6} - \frac{2\,d - \frac{d^2}{2} - \frac{3}{2} - \log_e d}{2\,(1-d)^3}\right] a^3 \qquad (19)$$

with subscripts 1 and 2, for symbols C, d, and a, to denote the appropriate end of the span. Similarly (see Table 1(e)), to find the statical moment of the $\frac{M}{E\,I}$-diagram for uniform load:

$$C = -\frac{1}{128} + \left[\frac{1}{8} - \frac{\frac{d}{2} - 1 + \frac{1}{2\,d}}{4\,(1-d)^2}\right] a^2 + \left[-\frac{1}{4} + \frac{-\frac{3\,d^2}{2} + 6\,d - \frac{9}{2} - 3\log_e d}{4\,(1-d)^3}\right] a^3 + \left[\frac{1}{8} - \frac{1 + \frac{3\,d}{2} - 3\,d^2 + \frac{d^3}{2} + 3\,d\log_e d}{2\,(1-d)^4}\right] a^4 \qquad (20)$$

Fixed-End Moments for Triangular Loads

The constants C were found to have different values when the small end of the triangular load was at the end farthest from the centroidal axis (see end 1, Fig. 1) than when the larger end was so placed. When the small end of the triangular load was at end 1, the constants were found to be as follows:

For the area of the $\dfrac{M}{E\,I}$-diagram (see Table 1(f))—

$$C_1 = \left[\frac{1}{3\,(1-d)^2}\left(0.5\,d - 1 + \frac{0.5}{d}\right) - \frac{1}{6}\right] a^2 - \left[\frac{1}{3\,(1-d)^4}\,(1 + 3\,d \log_e d + 1.5\,d - 3\,d^2 + 0.5\,d^3) - \frac{1}{12}\right] a^4 + \frac{7}{192} \qquad \text{(21a)}$$

and (see Table 1(h))—

$$C_2 = \left[\frac{2}{3\,(1-d)^2}\left(0.5\,d - 1 + \frac{0.5}{d}\right) - \frac{1}{3}\right] a^2 + \left[\frac{1}{(1-d)^3}\,(0.5\,d^2 - 2\,d + 1.5 + \log_e d) + \frac{1}{3}\right] a^3 + \left[\frac{1}{3\,(1-d)^4}\,(1 + 3\,d \log_e d + 1.5\,d - 3\,d^2 + 0.5\,d^3) - \frac{1}{12}\right] a^4 + \frac{3}{64} \qquad \text{(21b)}$$

For the statical moment of the $\dfrac{M}{E\,I}$-diagram (see Table 1(g))—

$$C_1 = \left[\frac{1}{6\,(1-d)^2}\left(0.5\,d - 1 + \frac{0.5}{d}\right) - \frac{1}{12}\right] a^2 + \left[\frac{1}{3\,(1-d)^3}\,(0.5\,d^2 - 2\,d + 1.5 + \log_e d) + \frac{1}{9}\right] a^3 - \left[\frac{1}{6\,(1-d)^4}\,(1 + 3\,d \log_e d + 1.5\,d - 3\,d^2 + 0.5\,d^3) - \frac{1}{24}\right] a^4 + \left[\frac{1}{3\,(1-d)^5}\,(0.5 - 4\,d + 4\,d^3 - 0.5\,d^4 - 6\,d^2 \log_e d) - \frac{1}{15}\right] a^5 + \frac{37}{5{,}760} \qquad \text{(22a)}$$

and (see Table 1(i))—

$$C_2 = \left[\frac{1}{3\,(1-d)^2}\left(0.5\,d - 1 + \frac{0.5}{d}\right) - \frac{1}{6}\right] a^2 + \left[\frac{7}{6\,(1-d)^3}\,(0.5\,d - 2\,d + 1.5 + \log_e d) + \frac{7}{18}\right] a^3 + \left[\frac{7}{6\,(1-d)^4}\,(1 + 3\,d \log_e d + 1.5\,d - 3\,d^2 + 0.5\,d^3) - \frac{7}{24}\right] a^4 - \left[\frac{1}{3\,(1-d)^5}\,(0.5 - 4\,d + 4\,d^3 - 0.5\,d^4 - 6\,d^2 \log_e d) - \frac{1}{15}\right] a^5 + \frac{53}{5{,}760} \qquad \text{(22b)}$$

Symbols a and d carry the subscript 1 in Eqs. 21a and 22a and the subscript 2 in Eqs. 21b and 22b. These subscripts are omitted from Eqs. 21 and 22 to simplify the typography. Eqs. 21 and 22 can be adapted for the case when the large end of the triangular load is at end 1 by reversing the subscripts. Constants C_1 and C_2, then, for the area of the $\frac{M}{E\,I}$-diagram are read from Tables 1(h) and 1(f), respectively; and for the statical moment of the $\frac{M}{E\,I}$-diagram from Tables 1(g) and 1(i), respectively.

Example 1

Find the stiffness and carry-over factors for the beam shown in Fig. 4.

Given: $L = 25.00$ ft; $a_1\,L = 5.00$ ft; $a_2\,L = 2.50$ ft; $h = 1.50$ ft; $d_1\,h = 3.30$ ft; and $d_2\,h = 3.00$ ft.

Calculate: $a_1 = \frac{5.00}{25.00} = 0.20$; $a_2 = \frac{2.50}{25.00} = 0.10$; $d_1 = \frac{3.30}{1.50} = 2.20$; and $d_2 = \frac{3.00}{1.50} = 2.0$.

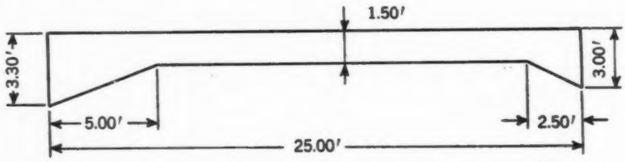


Fig. 4.

Area of Analogous Column.—From Table 1(a), using a_1 and d_1 to find C_1, and a_2 and d_2 to find C_2—$C_1 + C_2 = 0.3661 + 0.4375 = 0.8036$. Substituting this value in Eq. 1,

$$A = 0.8036\,\frac{L}{E\,I_o} \quad (23)$$

Centroid of Analogous Column.—From Table 1(b), using a_1 and d_1 to find C_1, and a_2 and d_2 to find C_2—$-\,C_1 + C_2 = -\,0.0690 + 0.0962 = 0.0272$. Substituting this value in Eq. 3,

$$Q_{\mathrm{I}} = 0.0272\,\frac{L^2}{E\,I_o} \quad (24)$$

By Eq. 7,

$$e_a = \frac{0.0272\,L^2}{E\,I_o} \div \frac{0.8036\,L}{E\,I_o} = 0.0338\,L \quad (25a)$$

$$L_1 = -\,\frac{L}{2} - e_a = -\,0.5338\,L \quad (25b)$$

and

$$L_2 = \frac{L}{2} - e_a = 0.4662\,L \quad (25c)$$

Second Moment of Analogous Column.—From Table 1(*c*), using a_1 and d_1 to find C_1, and a_2 and d_2 to find C_2—$C_1 + C_2 = 0.0179 + 0.0285 = 0.0464$. Substituting this value in Eq. 5,

$$Q_{II} = 0.0464 \frac{L^3}{E\,I_o} \qquad \text{(26a)}$$

and (Eq. 9)

$$Q_a = (0.0464 - 0.0009) \frac{L^3}{E\,I_o} = 0.0455 \frac{L^3}{E\,I_o} \qquad \text{(26b)}$$

Absolute Stiffness of End 1.—Substituting appropriate values in Eq. 11*a*,

$$K_1 = \frac{1.2444\,E\,I_o}{L} + \frac{6.2625\,E\,I_o}{L} = \frac{7.5069\,E\,I_o}{L} \qquad \text{(27)}$$

Relative Stiffness of End 1.—By Eq. 11*b*,

$$k_1 = \frac{7.5069\,E\,I_o}{L \times 4\,E} = \frac{1.8767\,I_o}{L} \qquad \text{(28a)}$$

Absolute Stiffness of End 2.—Substituting appropriate values in Eq. 12*a*,

$$K_2 = \frac{1.2444\,E\,I_o}{L} + \frac{4.7768\,E\,I_o}{L} = \frac{6.0212\,E\,I_o}{L} \qquad \text{(28b)}$$

Relative Stiffness of End 2.—Substituting appropriate quantities in Eq. 12*b*,

$$k_2 = \frac{6.0212\,E\,I_o}{L \times 4\,E} = \frac{1.5053\,I_o}{L} \qquad \text{(28c)}$$

Absolute Carry-Over Moment.—By Eq. 13,

$$N = \frac{1.2444\,E\,I_o}{L} + \frac{-\,5.4694\,E\,I_o}{L} = \frac{-\,4.2250\,E\,I_o}{L} \qquad \text{(29)}$$

Carry-Over Factors.—The carry-over factor away from end 1, by Eq. 14*a*, is $\frac{-\,4.2250\,E\,I_o}{L} \div \frac{7.5069\,E\,I_o}{L} = -\,0.563$; and, from end 2 (see Eq. 14*b*), is $\frac{-\,4.2250\,E\,I_o}{L} \div \frac{6.0212\,E\,I_o}{L} = -\,0.702$.

Example 2

Find the fixed-end moments for a uniform load totaling W pounds for the beam shown in Fig. 4. (Values for A, e_a, L_1, L_2, and Q_a are as taken from Example 1.)

Load on the Analogous Column.—From Table 1(*d*), using a_1 and d_1 to find C_1, and a_2 and d_2 to find C_2—$C_1 + C_2 = 0.0368 + 0.0405 = 0.0773$. Substituting this value in Eq. 15,

$$P = 0.0773 \frac{W\,L^2}{E\,I_o} \qquad \text{(30)}$$

Statical Moment of Load.—From Table 1(*e*), using a_1 and d_1 to find C_1, and a_2 and d_2 to find C_2—$-\,C_1 + C_2 = -\,0.0060 + 0.0073 = 0.0013$. Substi-

tuting this quantity in Eq. 16,

$$Q_f = \frac{0.0013\ W\ L^3}{E\ I_o} \qquad (31)$$

Then Eq. 17 yields

$$e_f = \frac{0.0013\ W\ L^3}{E\ I_o} \div \frac{0.0773\ W\ L^2}{E\ I_o} = 0.0168\ L \qquad (32a)$$

Finally,

$$e_f - e_a = (0.0168 - 0.0338)\ L = -\ 0.0170\ L \qquad (32b)$$

Fixed-End Moments.—Applying Eq. 18*a* to determine the fixed-end moment at end 1,

$$\text{FEM} = \frac{0.0773\ W\ L^2}{E\ I_o} \div \frac{0.8036\ L}{E\ I_o}$$

$$+ \frac{0.0773 \times -\ 0.0170}{0.0455} (-\ 0.5338) = 0.1116\ W\ L \qquad (33a)$$

Similarly, applying Eq. 18*b* to determine the fixed-end moment at end 2,

$$\text{FEM} = 0.0963\ W\ L + \frac{0.0773 \times -\ 0.0170}{0.0455} 0.4662 = 0.0828\ W\ L \qquad (33b)$$

EXAMPLE 3*a*

Find the fixed-end moment for a triangular load totaling W pounds, having the small end of the load at end 1, for the beam shown in Fig. 4. (Values of A, e_a, L_1, L_2, and Q_a are as taken from Example 1.)

Load on the Analogous Column.—From Table 1(*f*), using a_1 and d_1 to find C_1; and from Table 1(*h*), using a_2 and d_2 to find C_2—$C_1 + C_2 = 0.0319 + 0.0454 = 0.0773$. With these factors known, Eq. 15 becomes

$$P = \frac{0.0773\ W\ L^2}{E\ I_o} \qquad (34)$$

Statical Moment of Load.—From Table 1(*g*), using a_1 and d_1 to find C_1; and from Table 1(*i*), using a_2 and d_2 to find C_2—— $C_1 + C_2 = -\ 0.0047 + 0.0085 = 0.0038$. With these factors known, Eq. 16 becomes

$$Q_f = \frac{0.0038\ W\ L^3}{E\ I_o} \qquad (35)$$

Then, by Eq. 17,

$$e_f = \frac{0.0038\ W\ L^3}{E\ I_o} \div \frac{0.0773\ W\ L^2}{E\ I_o} = 0.0492\ L \qquad (36a)$$

and

$$e_f - e_a = (0.0492 - 0.0338)\ L = 0.0154\ L \qquad (36b)$$

Fixed-End Moments.—Applying Eq. 18*a* to determine the fixed-end moment at end 1,

$$\text{FEM} = \frac{0.0773\ W\ L^2}{E\ I_o} \div \frac{0.8036\ L}{E\ I_o}$$

$$+ \left(\frac{0.0773 \times 0.0154}{0.0455}\right) (-\ 0.5338) = 0.0822\ W\ L \qquad (37a)$$

Similarly, applying Eq. 18*b* to determine the fixed-end moment at end 2,

$$\text{FEM} = 0.0962\, W\, L + \frac{0.0773 \times 0.0154}{0.0455} 0.4662 = 0.1084\, W\, L \ldots (37b)$$

Example 3*b*

Find the fixed-end moment for a triangular load totaling W pounds, having the large end of the load at end 1, for the beam shown in Fig. 4. (Values of A, e_a, L_1, L_2, and Q_a are as taken from Example 1.)

Load on the Analogous Column.—From Table 1(*h*), using a_1 and d_1 to find C_1; and from Table 1(*f*), using a_2 and d_2 to find C_2—$C_1 + C_2 = 0.0417 + 0.0356 = 0.0773$. Eq. 15 then yields:

$$P = \frac{0.0773\, W\, L^2}{E\, I_o} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (38)$$

Statical Moment of Load.—From Table 1(*i*), using a_1 and d_1 to find C_1; and from Table 1(*g*), using a_2 and d_2 to find C_2—$C_1 + C_2 = -0.0067 + 0.0060 = -0.0007$. With these factors known, Eq. 16 becomes

$$Q_f = \frac{-0.0007\, W\, L^3}{E\, I_o} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (39)$$

Then, as in Example 3*a*, Eq. 17 yields

$$e_f = \frac{-0.0007\, W\, L^3}{E\, I_o} \div \frac{0.0773\, W\, L^2}{E\, I_o} = -0.0091\, L \ldots\ldots (40a)$$

and

$$e_f - e_a = (-0.0091 - 0.0338)\, L = -0.0429\, L \ldots\ldots (40b)$$

Summary

Published literature and texts present charts and tables whereby the design constants may be readily obtainable for structural members which have a haunch at only one end or which have haunches at each end that are symmetrical about the center of the span. Little information is available regarding these constants for beams which are nonsymmetrically haunched.

The usual method of obtaining the design constants for the nonsymmetrical case is laborious and time consuming.

This paper is presented to assist the engineer in obtaining the stiffness and carry-over factors for beams having nonsymmetrical straight haunches, and the fixed-end moments for uniform and triangular loads. The results were obtained by the use of the well-known column analogy. The labor required to compute the area, centroid, and moment of inertia of the analogous column, and the area and centroid of the $\frac{M}{E\, I}$-diagram has here been simplified so that they may be obtained by simple arithmetical combination of the constants presented in the tables. Examples have been included to illustrate the use of the tables and the procedure to be followed in obtaining the design constants.

The tables have not been extended to cover other loading conditions because of the great amount of work involved. The two cases covered are those most

commonly encountered and therefore the most important loading conditions to be considered. The tables can be enlarged to cover any desired loading condition and the procedure would be equally adaptable.

Acknowledgment

The primary source from which the material was assembled which deals with the analogous column and the calculation of the stiffness and carry-over factors was an academic thesis by the writer, entitled "Stiffness and Carry-Over Factors for Beams with Nonsymmetrical Straight Haunches," submitted to the University of Colorado, in Boulder, in 1939, in partial fulfilment of the requirements for the Degree of Master of Science in Civil Engineering. The part of the paper pertaining to the fixed-end moments has been prepared since that date.

APPENDIX

Notation

The following letter symbols, used in the paper, conform essentially to American Standard Letter Symbols for Mechanics, Structural Engineering and Testing Materials,[4] prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932:

A = area; total area of an analogous column;

a = ratio of the total span length L (Fig. 1):

$a_1 = a$ for distances $a_1 L$ measured from end 1;

$a_2 = a$ for distances $a_2 L$ measured from end 2;

C = a dimensionless constant selected from Table 1 to suit various conditions of a problem, the subscripts 1 and 2 designating the end of the beam; in Eq. 10 it is the distance from the center of gravity of the column section to the point at which the fiber stress is computed;

d = coefficient of depth h to indicate total depth of haunch:

$d_1 = d$ for depths $d_1 h$ measured at end 1;

$d_2 = d$ for depths $d_2 h$ measured at end 2;

E = modulus of elasticity of the material;

e = eccentricity = distance from the center of the span to the center of elastic weights (in Eq. 10 it is the eccentricity of the applied load P):

$e_a = e$ for an analogous column;

$e_f = e$ for the problem of fixed-end moments;

f = the fiber stress in the column formula, Eq. 10;

---

[4] ASA—Z10a—1932.

h = uniform depth of a beam without haunches and the minimum depth of a beam with haunches;

I = moment of inertia:

> I_o = inertia moment of the minimum section of a beam;
>
> I_x = inertia moment at a distance x along a haunch (Figs. 2 and 3);

K = absolute stiffness factor (Eqs. 11*a* and 12*a*) with subscripts 1 and 2 denoting the end of the beam;

k = relative stiffness factor (Eqs. 11*b* and 12*b*) with subscripts denoting the end of the beam;

L = length of a beam:

> L_1 = part of a beam to the left (end 1) of the gravity center of elastic weights (Fig. 2);
>
> L_2 = part of a beam to the right (end 2) of the gravity center of elastic weights;

M = bending moment;

N = a carry-over mom^nt induced at the far end of a member;

P = a load applied to an analogous column;

Q = statical moment of an area;

Q_{I} = first moment of the area of the analogous column about the center of the span (Eq. 3);

Q_{II} = second moment of the area of the analogous column about the center of the span (Eq. 5);

Q_a = total second moment of area of analogous column about the gravity axis (Eq. 9);

Q_f = statical moment of load acting on analogous column used in determining fixed-end moments (Eq. 16);

r = a carry-over factor (Eqs. 14), subscripts 1 and 2 being used to designate the ends;

W = total load on beam (Fig. 3);

x = variable distance from end 1 of a beam to a place on the haunch where the depth is y (Fig. 3);

y = depth of the haunch at any distance x from the left end of a beam (Fig. 3).

AMERICAN SOCIETY OF CIVIL ENGINEERS

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PAPERS

LOCK MANIFOLD EXPERIMENTS

By Edward Soucek,[1] Assoc. M. Am. Soc. C. E., and E. W. Zelnick,[2] Jun. Am. Soc. C. E.

Synopsis

In connection with the design of the Third Locks of the Panama Canal, a series of experiments dealing with the hydraulics of manifolds was performed in The Panama Canal Hydraulics Laboratory during 1939 and 1940. This paper presents a summary of the test results. Certain introductory and theoretical material which appeared desirable has been included. A method of predicting the performance of a manifold by the use of single port test data is presented and the method is checked by comparison with manifold tests. Some of the data are presented in abridged form but transcripts of original and more complete records have been filed with the Engineering Societies Library.[2a]

Introduction

Manifolds are important elements in the design of side-filling or bottom-filling locks. There is no generally accepted basis for their design, however. Discussions of a paper presented in 1940 by R. D. Gladding,[3] M. Am. Soc. C. E., and the absence of the subject from hydraulics textbooks are noteworthy in this connection. With few exceptions model tests used in the design of most recently built locks have been limited[4] to models of the entire lock. Consequently, there is little basis for designing the hydraulic system of locks that must meet special requirements or operate under unusual conditions. The size and importance of the new Panama Canal locks seemed to justify more fundamental treatment. The tests described herein were adopted to this end.

It is not easy to predict the time required to fill or empty a lock through a manifold system. It is customary to use a differential equation based on

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March 1, 1945.**

[1] Washington, D. C.

[2] With The Panama Canal, Balboa Heights, Canal Zone.

[2a] 29 West 39th St., New York, N. Y.

[3] "Loss-of-Head Determination in Uniformly Tapped Pipes," by R. D. Gladding, *Engineering News-Record*, November 21, 1940, p. 697; also discussions by F. W. Edwards, F. C. Zeigler, W. E. Howland, R. W. Powell, W. M. Langsford, and the author, *ibid.*, January 16, 1941, p. 78; April 10, 1941, p. 522; and May 22, 1941, p. 801.

[4] "The Manifold Problem in Lock Design," by Martin E. Nelson, *Bulletin 20, Proceedings of Hydraulics Conference*, Studies in Engineering, Univ. of Iowa, Iowa City, March, 1940, p. 160.

the orifice formula with an estimated coefficient, to compute the time required to fill or empty a lock under a falling head. Owing in part to complications resulting from inertial effects and to difficulty in allowing for time consumed by valve operations, but principally to lack of a satisfactory theory of manifold hydraulics, these attempts at simplification often have been unsuccessful.

Inequality in port discharges tends to defeat the purpose of a lock manifold.[5] These differences in discharge are produced by changes in pressure and velocity along the manifold tube. Elimination of the differences, or compensation for the differences, is possible by various methods, discussion of which is beyond the scope of this paper. It is evident that the use of such expedients as tapered culverts, varied port sizes, and varied port spacing is highly speculative without a specific model test of each design or some knowledge of what takes place within a manifold.

Purpose

The purpose of the tests was to provide data that would expedite the design of lock manifolds. Specifically, it was desired to give mathematical expression to the phenomenon of manifold flow in order to facilitate the original design and the modification of model manifolds. Since a single port can be built and tested quite easily and quickly, determination of a means of predicting the performance of a manifold on the basis of single port tests and verification of the technique by tests of a manifold were the principal phases of the program.

Outline

The first part of the study, "Theoretical Considerations," describes the mathematical approach adopted for the analysis. It is perhaps regrettable that several simplifying assumptions were necessary in order to obtain tangible results without unduly extending the test program. Substantial correctness for design purposes rather than precision was the reason for simplifying the theory to the greatest possible extent. Since disturbances within a lock chamber during emptying operations are usually (perhaps always) negligible in comparison with those that occur during filling, more attention was given to the discharge ports. The difference between filling and emptying conditions is due to the lower head-to-depth ratio which prevails during emptying operations and to the absence of what might be called jet action. The basic discussions deal with what will be called "discharge manifolds," "intake manifolds" being covered by brief supplementary statements.

The second part of the study, "Single Port Tests," covers the results of tests of seven different ports, each attached to a 6-in. square conduit. The ports were tested as both "discharge ports" and as "intake ports."

The third and final part of the study, "Verification Tests," describes tests of a manifold containing five identical ports which were duplicates of one of those tested as a single port. A 6-in. square conduit again served as the manifold tube. Observations are compared with calculations based on single port characteristics which had been determined previously.

[5] "Manual on Lock Valves," *Manual of Engineering Practice No. 3*, Committee on Lock Valves, Waterways Div., Am. Soc. C. E., 1930.

NOTATION

The following letter symbols, adopted for use in this paper, conform essentially to American Standard Letter Symbols for Hydraulics,[6] prepared by a Committee of the American Standards Association, with Society participation, and approved by the Association in 1942:

A = area:
 A_c = cross-sectional area of conduit, in square feet;
 A_p = throat area of the port, in square feet;
C = discharge coefficient in orifice formula with subscript d or i denoting discharge ports or inlet ports, respectively;
g = gravitational acceleration, in feet per second per second;
H = effective head on a given port, in feet, with subscript d or i denoting discharge ports or inlet ports, respectively;
k = a dimensionless coefficient to be studied experimentally, with subscript d or i denoting discharge ports or inlet ports, respectively;
n = Manning's roughness coefficient;
P = a ratio of discharges:
 $P_d = Q_3/Q_1$;
 $P_i = Q_3/Q_2$;
Q = discharge, in cubic feet per second:
 Q_1, in the conduit upstream from a port;
 Q_2, in the conduit downstream from a port;
 Q_3, through a port;
V = mean conduit velocity, in feet per second:
 V_1, upstream from port;
 V_2, downstream from port;
Z = elevation of the hydraulic gradient in feet; Z_1 = upstream elevation and Z_2 = downstream elevation; ΔZ = difference between upstream and downstream elevations;
γ = specific weight of water, in pounds per cubic foot.

THEORETICAL CONSIDERATIONS

It is evident that any fundamental study of the hydraulics of manifolds must be based upon an analysis of the behavior and effects of a single lateral port attached to a tube. Preliminary study made it clear that two problems were of primary importance—namely, (*a*) the pressure change which occurs in a tube in the vicinity of a lateral port and (*b*) the discharge through a lateral port and the factors that modify it. Solution of these two problems permits analysis of the hydraulics of a manifold by a simple computation procedure which resembles somewhat the calculation of a backwater curve. Mutual interference between closely-spaced consecutive ports is the principal limitation on the applicability of this procedure which otherwise appears quite general.

(*a*) *Pressure Changes.*—Consider a horizontal frictionless tube to which a discharging lateral port is attached. The port outlet is submerged in still water. The elevation of the water surface over the port outlet is called the

[6] ASA—Z10.2—1942.

"submergence level." The tube is connected to a source of supply which is at some elevation greater than the submergence level. Downstream from the port, the tube is throttled in such a manner that the hydraulic gradient past the port is above the submergence level. The port is of such a length that the discharge through it occurs substantially at right angles to the axis of the manifold tube. Since the tube is frictionless, the pressure gradients both upstream and downstream from the port are horizontal but at different elevations. In the immediate vicinity of the port, there is an abrupt pressure rise in the direction of flow.

Consider the rate of change of momentum that occurs between two sections on the horizontal pressure gradients upstream and downstream, respectively, from the port. Neglecting nonuniform velocity distribution, the rate of change of momentum has the value $\frac{Q_2}{g}\gamma\,(V_1 - V_2) + \frac{Q_3}{g}\gamma\,V_1$. The forces producing this effect are the pressure rise in the conduit and the resultant of unbalanced pressures within the port. The latter force is unknown and undeterminable in any practical manner except as a residual. The unbalanced port pressure is intimately related to the momentum change of Q_3 and, therefore, it seems likely that any allowance for its effect can be applied to the term representing that momentum change.

It is considered that k_d is that fraction of the momentum change of Q_3 which is produced by a pressure rise in the conduit. Conversely, $(1 - k_d)$ is that fraction of the momentum change of Q_3 which is produced by unbalanced pressure within the port. This line of reasoning leads to an expression involving the pressure rise, an equation which is not obtainable without some simplifying assumption similar to the one which has been explained. The equation obtained is:

$$\gamma\,A_e\,(Z_2 - Z_1) = \frac{Q_2}{g}\gamma\,(V_1 - V_2) + \frac{k_d\,(Q_3\,\gamma\,V_1)}{g} \quad \text{.........(1)}$$

By use of the law of continuity and some algebraic manipulation, Eq. 1 may be reduced to the form,

$$\frac{\Delta Z}{(V_1)^2/2\,g} = 2\,P_d\,(1 + k_d) - 2\,(P_d)^2 \quad \text{...............(2)}$$

Certain published data indicate that, for a given port and conduit, the ratio on the left side of Eq. 2 is a function of P_d only. . For this to be true, k_d must be either a constant or a function of P_d only. Fig. 1(*a*) is a comparison of tests by the U. S. Bureau of Reclamation[7] as interpreted by R. W. Powell,[8] M. Am. Soc. C. E., tests by John A. Oakey,[9] Assoc. M. Am. Soc. C. E., and the conclusions, supported by tests, of M. L. Enger, M. Am. Soc. C. E., and M. I. Levy.[10] The approximate diameter ratios, lateral to conduit, for the

[7] "Model Studies of Penstocks and Outlet Works," *Bulletin 2*, Pt. VI, Boulder Canyon Project Final Repts., Bureau of Reclamation, U. S. Dept. of the Interior, 1938.

[8] "Hydraulics of Sprinkling Systems for Irrigation," by J. E. Christiansen, *Transactions*, Am. Soc. C. E., Vol. 107 (1942), p. 221.

[9] "Hydraulic Losses in Short Tubes Determined by Experiments," by John A. Oakey, *Engineering News-Record*, June 1, 1933, p. 717.

[10] "Pressures in Manifold Pipes," by M. L. Enger and M. I. Levy, *Proceedings*, A.W.W.A., May, 1929, p. 659.

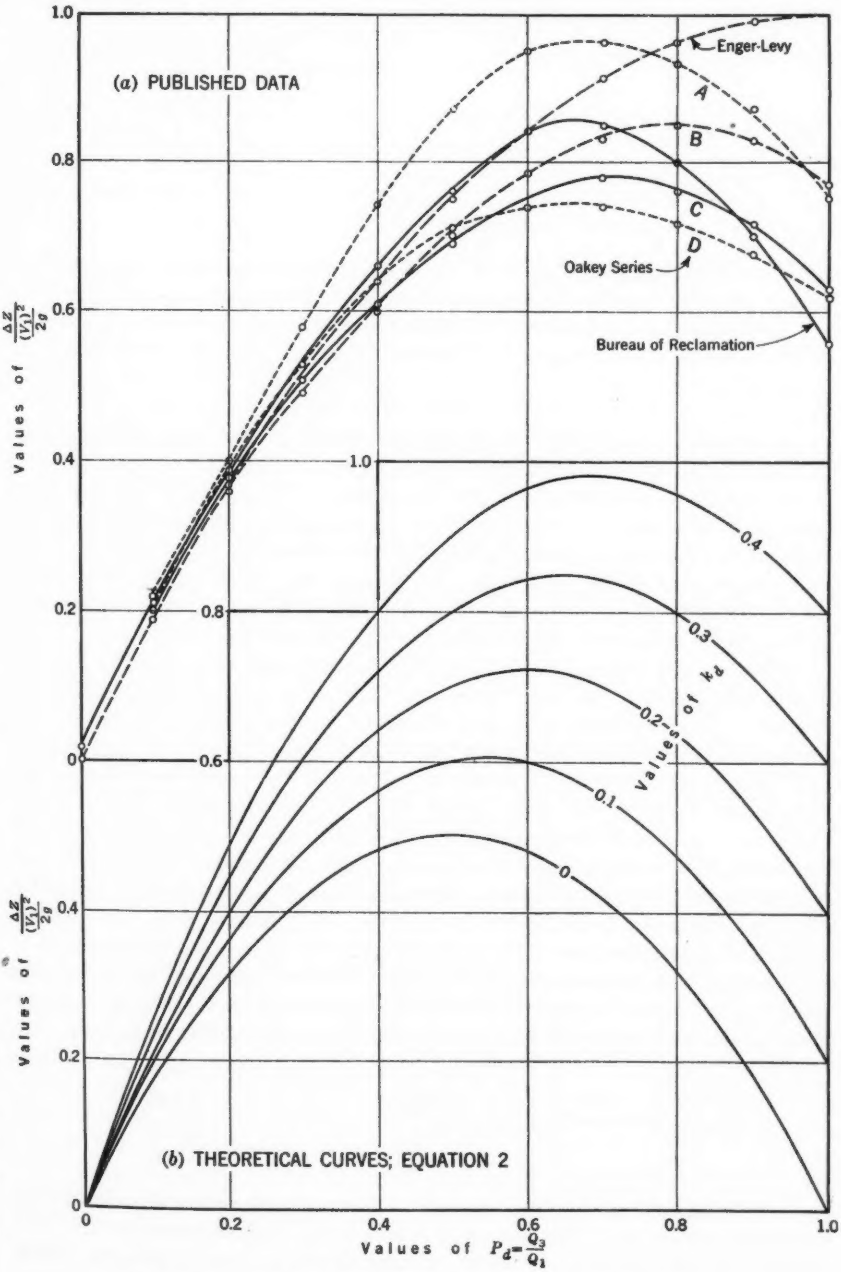

FIG. 1.—PRESSURE RISE; THEORETICAL EQUATIONS AND OBSERVED DATA

tests in Fig. 1(*a*) are as follows:

Series (see Fig. 1)	Diameter ratios
A	1:4.24
B	1:2.82
C	1:1.82
D	1:1.21
Enger and Levy	1:5.3
Bureau of Reclamation	1:2.3

In Fig. 1(*b*) are plotted graphs of Eq. 2 with various constant values assigned to k_d. It is evident that this comparison tends to support the analysis.

The published data exhibit a maximum relative pressure rise in the vicinity of P_d equal to 0.7, corresponding in that respect to the theoretical curve for k_d equal to 0.4. However, the relative pressure rises are of lower magnitude than those indicated by the theoretical curve for k_d equal to 0.4.

It should be noted that the graphs of relative pressure rise plotted against P_d are simply a convenient means of evaluating the effect of a residual unknown —the resultant of unbalanced port pressures. The graphical method could have been evolved by some other line of reasoning. However, the method appears to represent an approximately correct interpretation of the mechanics of pressure changes in manifolds. Of course, the effects of nonuniform approach velocity distribution and of angularity of port discharge are reflected in the relations between the relative pressure rise and P_d.

Since no real conduit is frictionless, a method had to be devised for measuring the pressure rise for a real manifold so as to minimize the effect of friction. The gradients in a real manifold are not level but, beyond the immediate effects of the port, they are straight lines. In analysis and application of pressure rise data, it is considered that the pressure change is sudden and occurs opposite the port center line. In practice then, the upstream gradient is extended downstream and the downstream gradient is extended upstream, both at their respective friction slopes, until they intersect a vertical erected at the center of the port. The intercept on this vertical, between the extended gradients, is considered to represent the pressure rise. As a matter of fact, the pressure rise is exceedingly abrupt. The described method of interpreting the pressure rise makes the calculated gradient for a manifold consist of a series of sloping lines with a step at each port. Fig. 2 illustrates these relations, the discharges, in cubic feet per second, being as follows:

Flow	Discharge Port	Intake Port
Q_1	2.631	1.844
Q_2	2.222	2.245
Q_3	0.409	0.401

An intake port is analyzed similarly. For an intake port, the elevation of the water surface over the port inlet is called the "submergence level." The conduit pressure gradient is below, instead of above, the submergence level. The port discharges into the conduit rather than out of it. A drop in pressure, in place of a rise, occurs in the direction of flow. The result of reasoning

identical to that used for a discharge port is a similar equation,

$$\frac{\Delta Z}{(V_2)^2/2\,g} = 2\,P_i\,(1 + k_i) - 2\,(P_i)^2 \quad \text{.........(3)}$$

It is evident from physical considerations that k_i will be larger than k_d. Since water enters the port from a quiescent pool, the unbalanced port pressure is

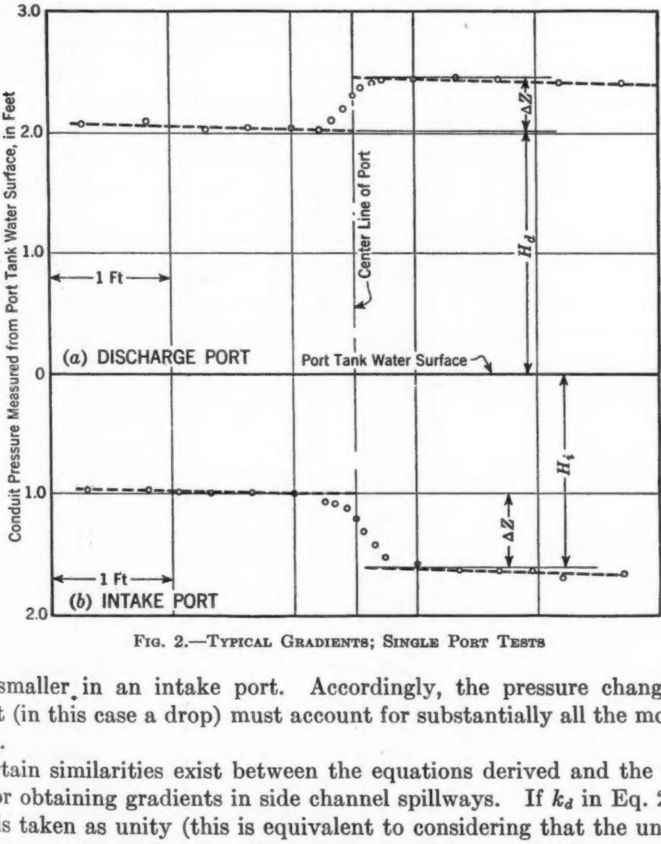

FIG. 2.—TYPICAL GRADIENTS; SINGLE PORT TESTS

much smaller in an intake port. Accordingly, the pressure change in the conduit (in this case a drop) must account for substantially all the momentum change.

Certain similarities exist between the equations derived and the formulas used for obtaining gradients in side channel spillways. If k_d in Eq. 2 or k_i in Eq. 3 is taken as unity (this is equivalent to considering that the unbalanced port pressure is zero), the value of pressure change obtained—actual, not relative—will equal twice the difference between the velocity heads upstream and downstream from the port. It is evident, of course, that Bernoulli's theorem is not applicable to the phenomenon under consideration, although a modified energy equation might be used.

(*b*) *Port Discharge.*—After considering various methods, it was decided to attempt to relate the port discharge to some head measurement by the use of the simple orifice equation ($Q_3 = C\,A_p\,\sqrt{2\,g\,H}$), and then to study variations in the resulting "discharge coefficient." Although it could not have been predicted definitely that this approach would be successful, study of published data[4,9,10] indicated that it was promising.

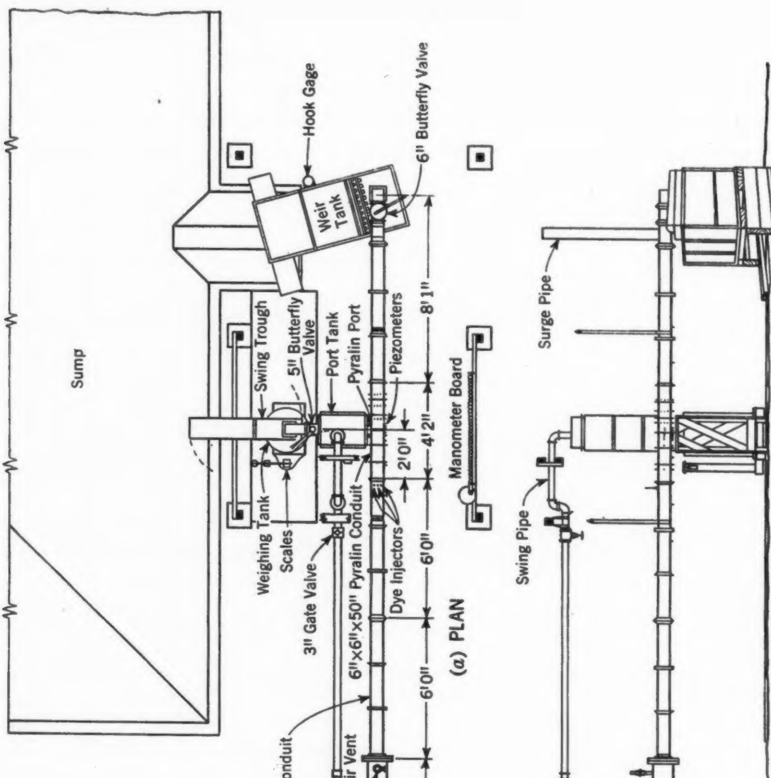

Fig. 3.—Single Port Test Apparatus

The hydraulic gradient in the vicinity of a port changes so rapidly that there is no natural method of defining the "head on the port." Since the choice is arbitrary, the head is defined in a manner which seems definite, convenient, and logical. The elevation difference between the submergence level and the intersection of the upstream gradient with a vertical line erected opposite the center of the port (the same intersection previously used in evaluating pressure change) is defined as the effective head, H_d, on a discharge port. This relationship is illustrated in Fig. 2.

For an intake port, the effective head, H_i, is defined as the elevation difference between the submergence level and the intersection of the downstream gradient with the intersection of a vertical line erected opposite the center of the port. This relationship is also shown in Fig. 2.

It might appear desirable to add the head corresponding to the mean velocity of approach to the effective head for a discharge port and to make a corresponding adjustment to the effective head for an intake port. This was not done for several reasons. The method of measuring the effective head and the discharge equations are rather arbitrary so it seemed desirable to use simple definitions. The principal effect of high conduit velocity approaching the port, considering now a discharge port, would appear to be such as to reduce the port discharge. Accordingly, it seemed illogical to insert a velocity term into the discharge equation in a manner that would indicate the contrary. The effect of conduit velocity is considered separately in studying the variation in discharge coefficients. Certain minor complications which are due to the manner in which the effective head is defined are discussed in another section of the paper.

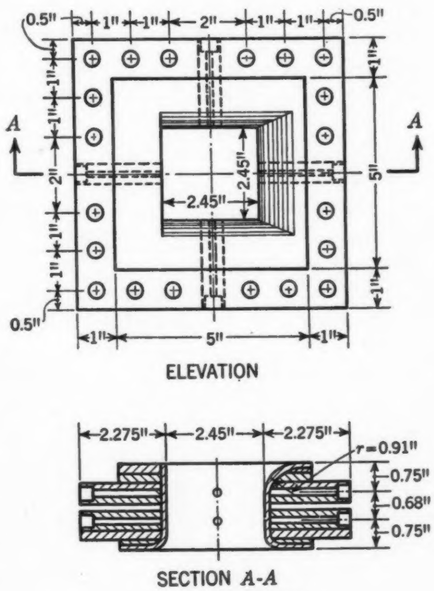

Fig. 4.—Port VII

Single Port Tests

Apparatus.—The arrangement of test apparatus is shown in Fig. 3. The equipment was designed to simulate the range of heads, velocities, and port and culvert areas which ordinarily occur in lock manifolds. Port VII, shown in Fig. 4, was placed in the conduit with square corner vertical and the large radius opposite the square corner on the upstream side of the port for the discharge tests. The large radius was on the downstream side of the port for the intake tests. All verification tests were made with Port VII. Radii that are not shown in Fig. 4 are 0.36 in.

Ports I to VI, inclusive (not illustrated), had the dimensions shown in Table 1. The ports were equipped with flanges that fitted into openings in the conduit and port tank, so arranged that the radii of the rounded ports were tangential to the tank and conduit faces; also, the faces of the sharp-cornered ports were in the planes of the tank and conduit faces.

TABLE 1.—DIMENSIONS OF PORTS I TO VI

Port	Length (in.)	Height and width at throat (in.)	Top and bottom at ends of port	Sides at ends of port
I	3.50	3.50	Square	Square
II	3.50	2.75	Square	Square
III	3.50	2.00	Square	Square
IV	3.50	3.50	0.75[a]	1.25[a]
V	3.50	2.75	0.59[a]	0.98[a]
VI	3.50	2.00	0.43[a]	0.71[a]

[a] Radius of edge fillet, in inches.

In order to obtain parallel flow upstream from the port, a set of vanes was placed in the conduit as shown in Fig. 3. To simulate a discharge port, all the flow was supplied by the 10-in. line. To simulate an intake port, the port discharge was supplied by the 3-in. auxiliary line. Discharges and pressures were controlled by the valve on the 10-in. pipe, the valve on the 3-in. swing pipe, the butterfly valve on the port tank outlet, and the butterfly valve at the end of the conduit.

Particular care was taken in the construction of the pyralin conduit to prevent bulging under pressure. All piezometers except those in the port tank were placed in the pyralin sections where greater case in installation and maintenance was possible.

Flow patterns were observed by injecting dye through the piezometer connections. For some of the tests, velocities at various cross sections were obtained with a pitot tube, inserted through special openings provided for that purpose.

The level in the port tank was determined from the piezometers connected to the bottom of the tank at several points. The baffle shown in Fig. 5 was placed in the port tank to prevent excessive turbulence and entrainment of air during the intake tests. Port discharge measurements were made by diverting the flow into a 50-gal weighing tank. Conduit discharges were measured with a calibrated 90° V-notch weir.

Test Procedure.—For each port, intake and discharge tests were made with effective heads, previously defined, ranging from 0.3 to 3.0 ft. In addition to the variations in head, the conduit discharges were varied from zero to 2.5 cu ft per sec downstream from the port.

The hydraulic gradient in the culvert, the level in the tank, and the pressures in the port were obtained by reading forty-five manometers ten times during the course of each test and averaging the results for each manometer. In discharge port tests, discharge measurements for the port and the culvert were made at the beginning of each test, during the test, and at the conclusion of the test. For the intake port tests, discharge measurements were made

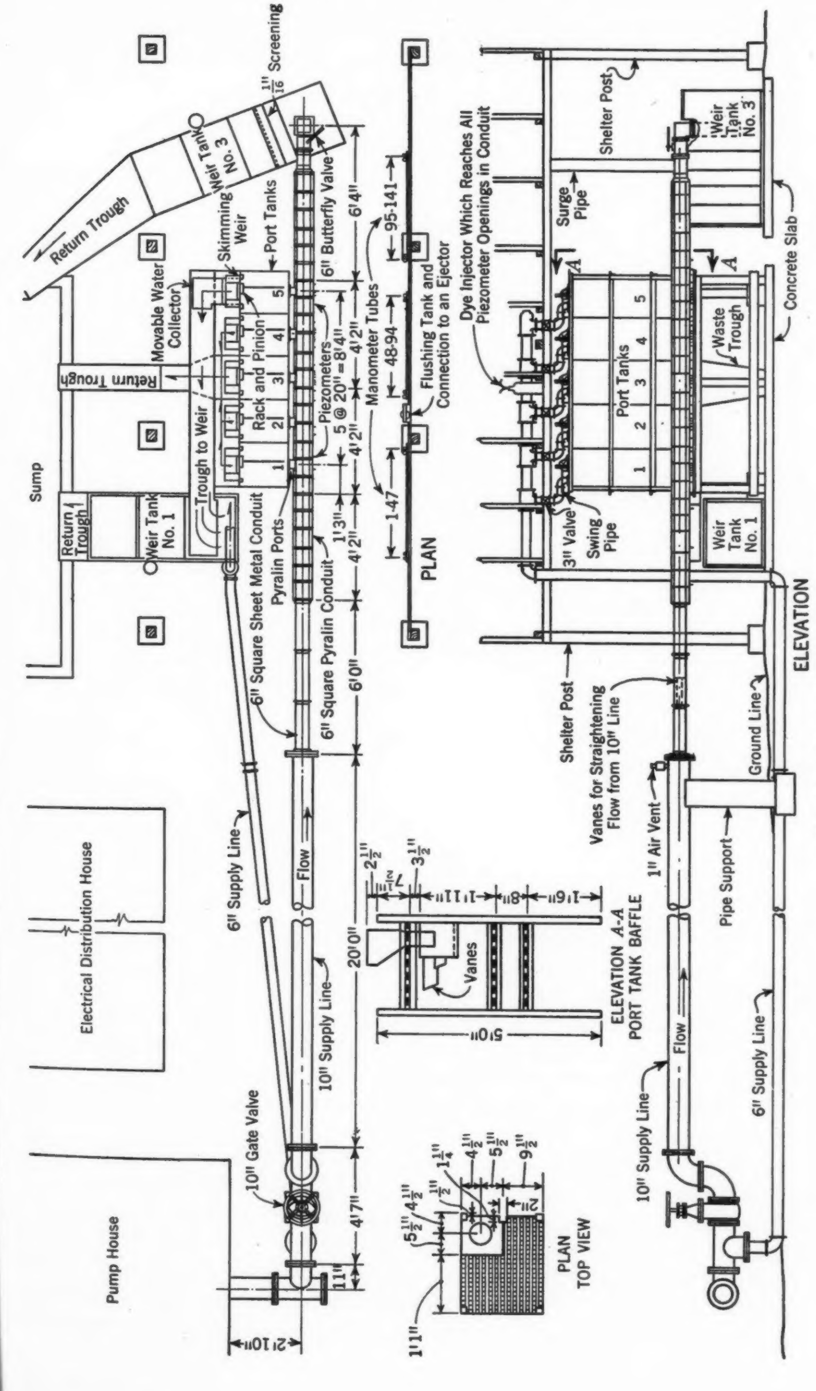

Fig. 5.—Manifold Test Apparatus

only at the beginning and at the end of each test, since the port discharge had to be diverted for measurement, thus disturbing the test conditions.

Several tests were made at various times to determine and check the roughness of the pyralin conduit. With undisturbed flow, friction slopes could be approximated closely by using an n of 0.009 in Manning's formula.

Analysis.—It proved possible to reduce the data, along the lines suggested under "Theoretical Considerations," to four characteristic curves for each port. Fig. 6 shows the characteristic curves for Ports I to VI, inclusive. To save space, these curves are shown to a very small scale and observation

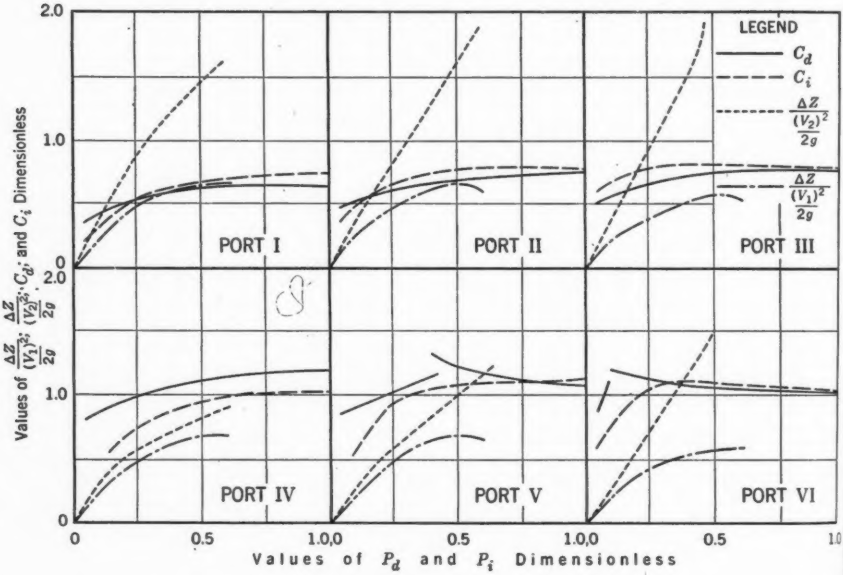

FIG. 6.—PORT CHARACTERISTICS, PORTS I TO VI, INCLUSIVE

points are omitted. The complete data for Ports I to VII, to be used by interested investigators in plotting the complete set, are filed with the Engineering Societies Library. Fig. 7 shows the characteristic curves for Port VII to a larger scale with observations indicated. The curves for Port VII are indicative of the agreement between curves and observation points for Ports I to VI, inclusive. Since each observation point represents a ratio of two observed quantities—some of which are very small—considerable "scatter" is to be expected.

This method of analysis was adopted for a number of reasons. The tentative nature of the theory did not justify a complex treatment. The relative pressure changes and discharge coefficients showed no significant dependence upon variables other than P_d and P_i. The time available for testing was limited; the need for tangible results great. The amount of data required for the method of analysis adopted was relatively small. The range of conditions under which lock manifolds operate makes certain compromises

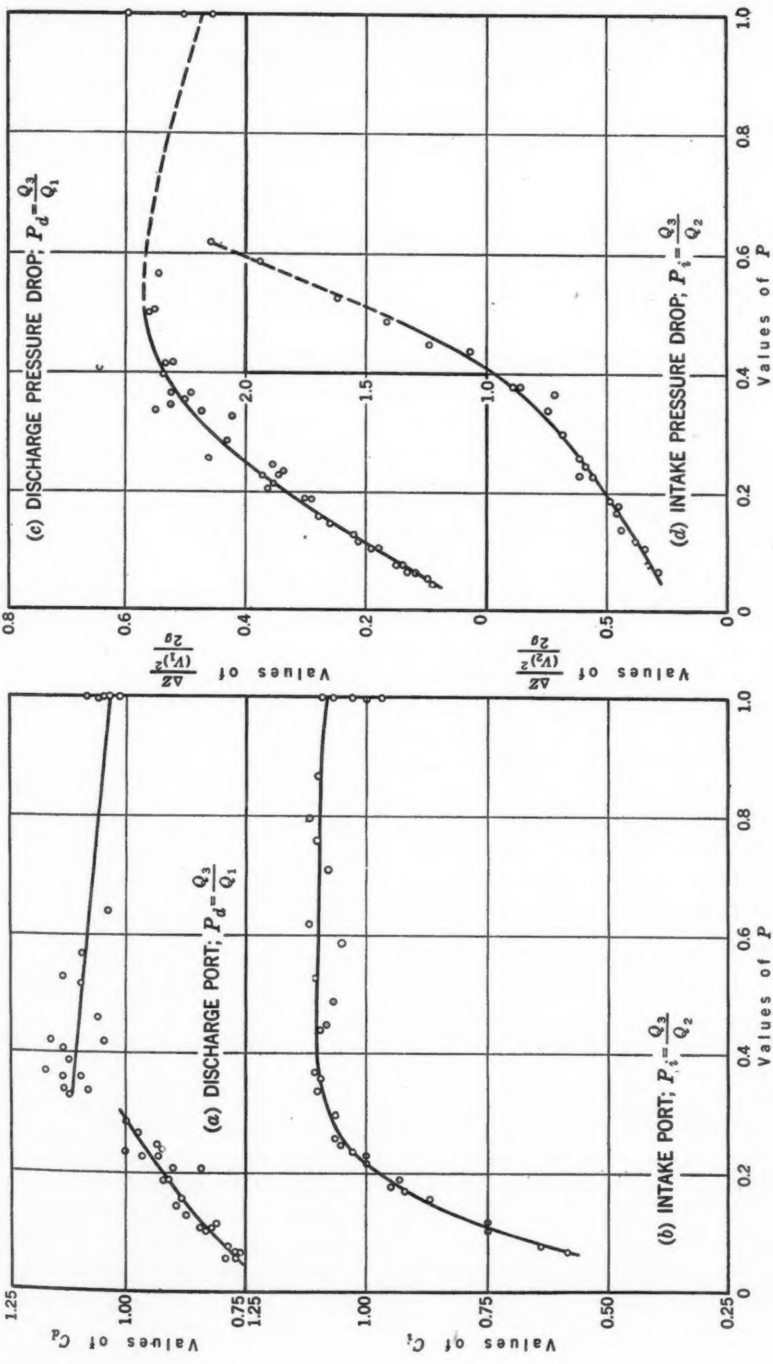

FIG. 7.—PORT CHARACTERISTICS, PORT VII

in design inevitable. Inertial effects make real precision nearly impossible to attain. The dimensionless treatment makes it possible to solve many problems by proportion and simplifies model-prototype ratios. Finally, and most important, there is a reasonable physical basis for the relations used. Similar flow patterns and, therefore, similar relations between physical quantities might be expected to prevail at given values of P_d and P_i for a particular port, regardless of the absolute magnitude of the quantities involved.

The port discharge is a factor that occurs to the first power in both coordinates of the coefficient curves. Therefore, some other combination of variables, omitting the port discharge as a factor, could have been used. This apparent simplification did not prove advantageous so the more significant combinations were retained.

Discussion of Results.—The limitations of the apparatus made it impossible to obtain large values of ΔZ simultaneously with large values of P_d or P_i. Only the farthest downstream port in a discharge manifold and the farthest upstream port in an intake manifold have values of P_d and P_i, respectively, exceeding about 0.5. The pressure in the closed end is not required in analysis. It was not justifiable, therefore, to build the apparatus so as to reduce observational error in the pressure change when P_d or P_i exceeded about 0.5. The observed values in the unreliable range are only a few hundredths of a foot in magnitude and large percentage errors are inevitable. The use of ratios in the graphs does not show the absolute magnitude of the pressure changes and, when the pressure change itself is small, a very small observational error produces a large effect on the relative pressure change. For this reason, the pressure-change curves are not considered to be established beyond P_d and P_i exceeding 0.6. However, the data filed with the Engineering Societies Library show the observations beyond the range of the curves.

The rounded ports, particularly the smaller sizes, exhibit a discontinuity in the relations between C_d and P_d. The largest rounded port (Port IV) probably has a similar but minor discontinuity at a value of P_d equal to about 0.75. From observation of dye streams, the discontinuity is believed to mark the value of P_d at which a change in the character of the flow through the discharge port occurs. An eddy which existed along the upstream side of the port for values of P_d less than that corresponding to the discontinuity was not evident for higher values. Tests in the vicinity of the discontinuity were difficult to establish and conditions tended to be unstable. Fortunately, the magnitude of the break in the relation was not sufficient to produce serious error in the calculations.

With the exceptions to which attention has been called, the characteristics of the data are fairly consistent and regular. Although the tests are evenly divided between effective heads of 0.3, 0.6, 1.0, 1.5, 2.0, 2.5, and 3.0 ft, the data for each port were found to be represented with fair accuracy by the dimensionless curves. The effect of rounded port entrances is evident and as expected. The effect of port size or, by inference, port area to culvert area ratio is reasonably well established over a large range and probably can be extrapolated somewhat.

Limitations.—Before explaining the application of the data to manifold calculations, it seems desirable to call attention specifically to certain limitations. All characteristic curves for the seven ports are based on single port observations. Approach conditions were normal for a smooth conduit. Downstream from the port, the velocity distribution was irregular. Although the pressure change was abrupt, as shown by the typical gradients of Fig. 2, the velocity distribution remains irregular for some distance downstream from a port. The slope of the hydraulic gradient was found, in the single port tests, to reach a value which was normal for the mean velocity within a very short distance from the port. This condition does not necessarily indicate a normal velocity distribution. A gradual reduction in the velocity head coefficient might obscure a tendency toward a greater than normal slope produced by the abnormal flow pattern. Although it is frequently assumed that the excess energy due to a distorted velocity distribution is not recoverable, the implication of a normal friction slope cannot be accepted without reservations. Accordingly, although it is believed that the observed pressure changes correctly represent the displacement between upstream and downstream gradients for values of P_d and P_i below about 0.5, it is not claimed that the disturbed flow pattern may not affect the performance of an adjacent port. On the other hand, the relatively small rate of change in discharge coefficient with P_d or P_i indicates that the effect of velocity distribution upon port discharges will not be large.

It has been observed that some discharge may occur through a port when the head, as defined, is negative. Negative heads could have been avoided by introducing a velocity head term into the effective head and this may ultimately prove to be the best solution. The discharges accompanying negative heads, as herein defined, tend to be irregular and of small magnitude. In general, it is felt that this condition can and should be avoided in design. For very low values of P_d and P_i, however—below the range covered by the tests—the method of analysis would cause the coefficient curves to rise rapidly as the zero value of P_d or P_i is approached. The curve of discharge coefficients should not be extrapolated to low values of P_d or P_i.

The data are based on tests of ports of particular sizes, shapes, and lengths—attached to a conduit of a particular size and shape. In view of the absence of data on other shapes and sizes, it is likely that some extrapolation from these tests will be attempted. If done carefully, such extrapolation should give better results than a guess or empirical rule. Strictly, however, the results are applicable only to geometrically similar designs. Any other application should be made with the realization that substantial errors are possible.

Application.—The design of a manifold is an indirect process. Characteristic curves for the ports used must be established by single port tests or otherwise. A manifold is assumed and analyzed. If the result is unsatisfactory, the design must be modified. This procedure is repeated until the required relations are obtained. Only the analysis of a manifold with a series of identical ports will be considered. Other applications are obvious.

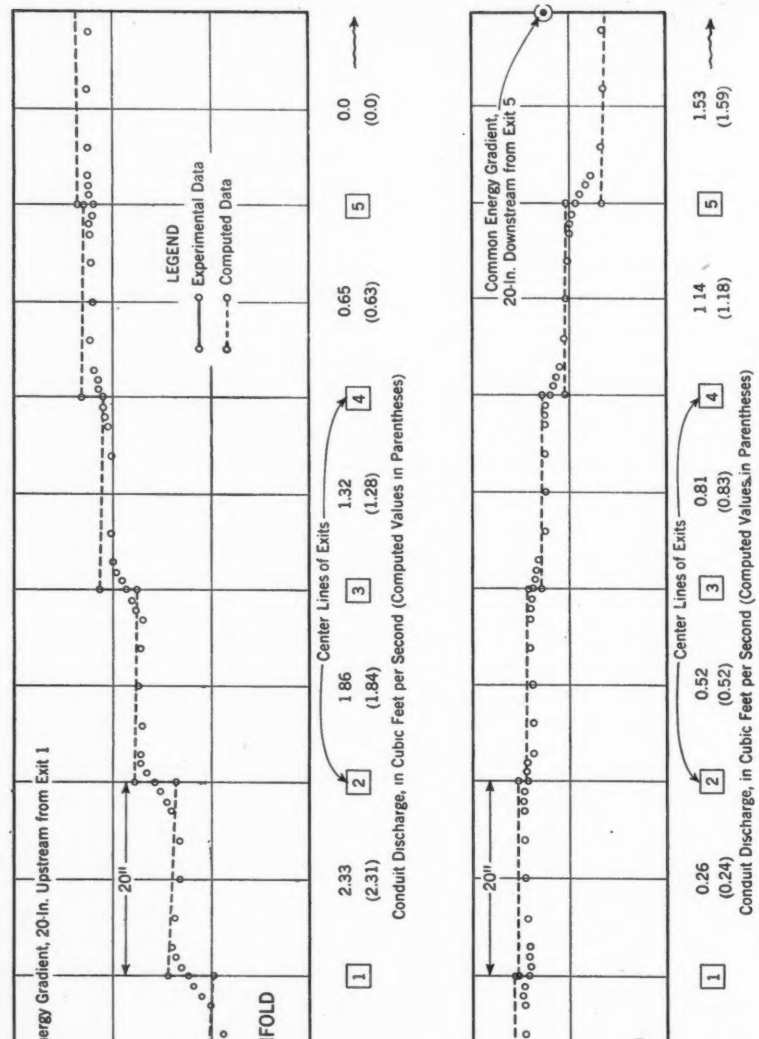

FIG. 8.—COMPARISON OF CONDUIT PRESSURES AND DISCHARGES, PORT VII
(Pressure Measured from the Water Surface in the Port Tank)

Assume a discharging manifold. The farthest downstream port has a value of P_d equal to unity. Any convenient discharge through this port may be assumed and the head calculated, the value of C_d being known. The friction loss to the next port can be calculated, since the discharge is known, and added to the previously determined head, the result being the head immediately downstream from the next port. The discharge through the second port from the downstream end is then assumed and the heads upstream and downstream are calculated, P_d being determinable for any assumed discharge. Only one discharge can be found which is consistent with the established discharge through the end port and the resulting gradient. The calculation proceeds up the manifold by trial until the farthest upstream port is passed. Since friction losses may be considered to vary with the square of the mean velocity, the distribution of the total discharge is the same, regardless of the port discharge originally assumed. Accordingly, the calculation made for one discharge may be adjusted to any other head or discharge by the following rules: (1) All heads vary in proportion to the over-all head applied to the manifold; and (2) all discharges and velocities vary as the square root of the applied head. An example of this adjustment is shown under "Verification Tests."

If the calculation indicates a negative head upstream from any port, the method fails. However, this result indicates that the port area is too great

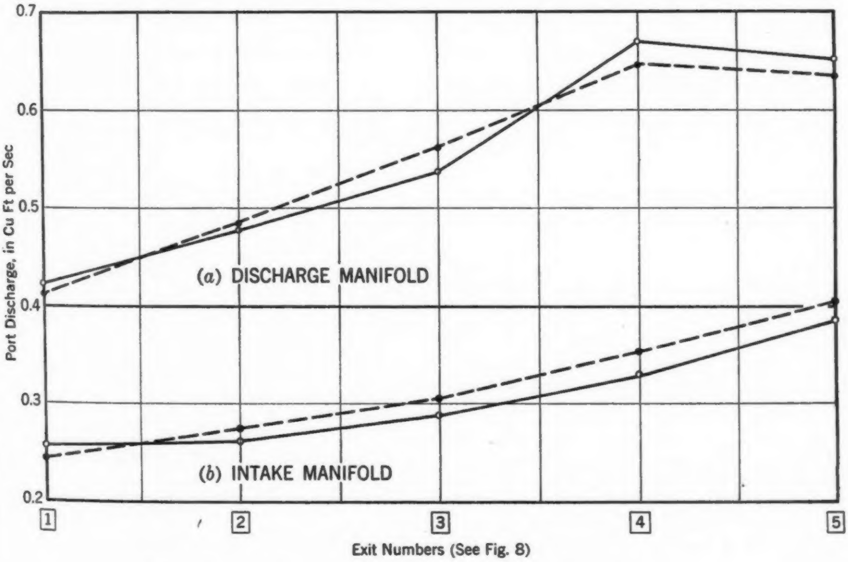

FIG. 9.—COMPARISON OF PORT DISCHARGE, PORT VII

for good distribution. This may be corrected by reducing the size, number, or efficiency of the ports or by increasing the size of the conduit. The choice between these adjustments is governed by the desired relation between the applied head and the corresponding discharge and by the required distribution.

Methods of adjusting the distribution of discharge along the length of the manifold by varied port size or spacing are obvious.

For an intake manifold, the procedure is similar to that which has been explained. The calculations begin with the port at the upstream end of the manifold and proceed downstream.

It is possible to make the calculation without assigning a numerical value to the port discharges but this method has not been found advantageous. Examples of the calculations are presented under "Verification Tests."

Verification Tests

Apparatus, Tests, and Procedure.—The apparatus used for the verification tests is shown in Fig. 5. Ports type VII were connected to the conduit at five points at a 20-in. spacing, each port leading to a separate tank. The water levels in the tanks were equalized by adjustment of slide valves for discharge tests and by regulating the inflow for intake tests. A small 90° V-notch weir was used to measure port discharges. For the intake tests, each port was supplied by a 3-in. swing pipe connected to a 6-in. auxiliary line. The test procedure was substantially as described for the single port tests.

Results.—Figs. 8 and 9 show typical comparisons of experimental pressures and discharges with calculated values. The basis of comparison is a common energy gradient 20 in. upstream from the farthest upstream port—for the discharge manifold—and 20 in. downstream from the farthest downstream port for the intake manifold. The number of significant figures used in the tabular calculations is not justified by the nature of the data and is used merely to illustrate better the method of making friction adjustments. Table 2 shows the calculations for the discharge manifold, based on a port discharge

TABLE 2.—Typical Calculations, Discharge Manifold

Exit No.	Q_2	Q_3	Q_1	P_d	$\frac{\Delta Z}{\frac{(V_1)^2}{2g}}$	ΔZ	C_d	H_d	H_f	H_d+H_f[a]	$H_d+\Delta Z$[a]
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
5	0.000	0.500	0.500	1.000	....	....	1.032	2.086	0.004	2.090	....
4	0.500	0.509	1.009	0.505	0.568	0.143	1.090	1.947	0.016	1.963	2.090
3	1.009	0.444	1.453	0.305	0.455	0.238	1.010	1.718	0.033	1.751	1.956
2	1.453	0.377	1.830	0.206	0.330	0.274	0.923	1.481	0.053	1.534	1.755
1	1.830	0.326	2.156	0.151	0.251	0.289	0.871	1.247	....	....	1.536

[a] When the value in Col. 11 of any row substantially corresponds with the value in the preceding row, the assumed discharges are verified.

of 0.5 cu ft per sec for the farthest downstream port. The energy head calculated for this case—$H_1 + \frac{(V_1)^2}{2g} + H_f$—was $1.25 + 1.15 + 0.07 = 2.47$ ft. In Figs. 8(*a*) and 9(*a*), the tabulated results have been adjusted to an energy head of 3.95 ft, which is the experimental value. For the intake manifold the computed energy head—$H_2 - \frac{(V_2)^2}{2g} + H_f$—was $1.43 - 0.66 + 0.04 = 0.80$.

As before, Figs. 8(*b*) and 9(*b*) show the tabulated results adjusted to an energy head of 0.78 ft, to agree with the experimental value. For this case Table 3, contains the corresponding computations, starting with a discharge of 0.25 cu ft per sec for the extreme upstream port. Since the theory indicates constant

TABLE 3.—TYPICAL CALCULATIONS, INTAKE MANIFOLD

Exit No.	Q_1 (1)	Q_3 (2)	Q_2 (3)	P_i (4)	$\frac{\Delta Z}{\frac{(V_2)^2}{2g}}$ (5)	ΔZ (6)	C_i (7)	H_i (8)	H_f (9)	$H_i + H_f$[a] (10)	$H_i + \Delta Z$[a] (11)
1	0.000	0.250	0.250	1.000	....	....	1.080	0.479	0.001	0.480	....
2	0.250	0.283	0.533	0.531	1.580	0.107	1.102	0.588	0.005	0.593	0.481
3	0.533	0.317	0.850	0.373	0.870	0.156	1.095	0.748	0.011	0.759	0.592
4	0.850	0.362	1.212	0.299	0.685	0.250	1.078	1.008	0.023	1.031	0.758
5	1.212	0.418	1.630	0.256	0.600	0.394	1.046	1.427	....	....	1.033

[a] When the value in Col. 11 of any row substantially corresponds with the value in the preceding row, the assumed discharges are verified.

distribution of total discharge between the ports for any applied head and, as all experiments confirmed this result, tests at other heads are compared with calculated values only on the basis of total discharges and energy heads. These comparisons are shown, for the discharge and intake manifolds, in Fig. 10.

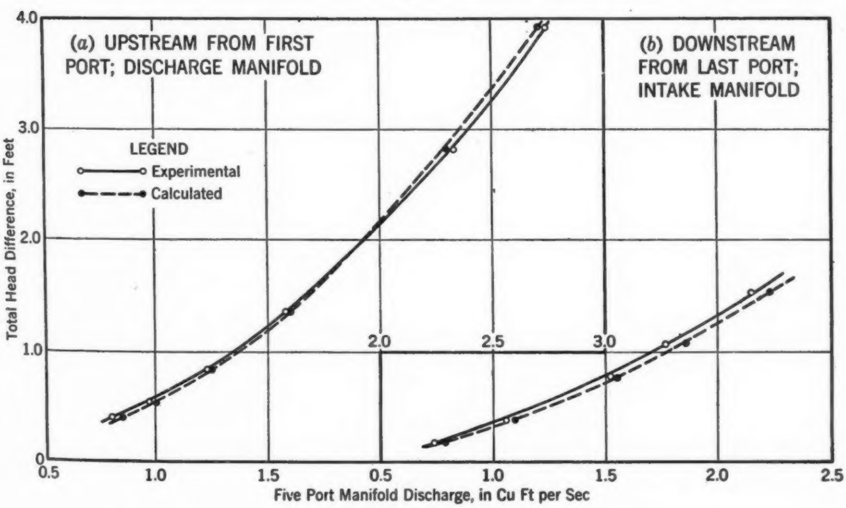

FIG. 10.—COMPARISON OF COMPUTED AND OBSERVED MANIFOLD DISCHARGES

CONCLUSION

The tests indicate that the application of single port test data to the design of manifolds for lock hydraulic systems is feasible. The use of dimensionless characteristic curves is a convenient generalization which expedites the analysis

and application of single port test data. The theory presented appears to provide a reasonably correct basis for the analysis of problems in manifold hydraulics; its imperfections and its merits are due largely to its simplicity.

Acknowledgment

F. W. Edwards, M. Am. Soc. C. E., preceded the senior writer as Chief of the Hydraulic Section of the Special Engineering Division. He initiated the manifold tests, took an active interest in the work, and reviewed the text of this paper. S. O. Steinborn, Jun. Am. Soc. C. E., conducted the verification tests. P. S. O'Shaughnessy reviewed the text. Several other members of the Hydraulic Section were employed on the tests and calculations. The cooperation of all is gratefully acknowledged.

The Special Engineering Division was directed by Maj. Gen. T. B. Larkin, Supervising Engineer, and Brig. Gen. Hans Kramer, Assistant Supervising Engineer, Members, Am. Soc. C. E. E. E. Abbott, Designing Engineer, and J. E. Reeves, Assoc. M. Am. Soc. C. E., Assistant Designing Engineer, were in charge of design.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

CONCENTRATION OF SEWAGE SLUDGE

A SYMPOSIUM

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March 1, 1945.**

EXPERIENCE IN LOS ANGELES, CALIF.

BY R. F. GOUDEY,[1] M. AM. SOC. C. E.

SYNOPSIS

Interesting research on the thickening of various types of sludge was conducted on a practical scale by the Bureau of Water Works and Supply of the City of Los Angeles at a 200,000 gal per day demonstration plant which operated continuously from May 12, 1930, to January 6, 1934.[2] The plant, treating a raw sewage averaging 388 ppm total suspended solids and 394 ppm biochemical oxygen demand, included units of preliminary sedimentation, activated sludge, final sedimentation, coagulation and filtration of various types of sludges, sludge digestion, coagulation and filtration of activated sludge effluent, sludge concentration, sludge digestion, and vacuum filtration of sludge. The plant was radically designed in that the aeration period was 3 hr and was radically operated in four respects: Dissolved oxygen was maintained in the settled activated sludge; aeration tank solids were maintained at 1,000 to 1,600 ppm; returned sludge was carried at 45% or less; and returned sludge was added at different stages in the aeration process.

INTRODUCTION

Several axioms regarding sludge volumes must be kept in mind: First, as percentage removals of sludge from plain settling tanks increase, so the percentage moisture in the sludge increases. Second, sewage solids passing preliminary clarifiers into aeration or chemical processes become quite liquid even up to 99.5%. This means that in a complete treatment plant the greater the percentage removal of sludge in the preliminary clarifier, the lower will be the total sludge volume. Third, sludge-bulking characteristics apply not only to activated sludge, but to fresh and digested sludge as well. Fourth, washing sludge in clarifiers from storm flows or clear night flow or diluting sludges with fresh water (which practices have been recognized and used even prior to 1930) tend to produce denser sludges.

SLUDGE BULKING

Bulking of sludge is a phenomenon not restricted solely to activated sludge. For a given sewage, the fresh sludge moisture content increases with the percentage removal of total solids. Fresh sludge itself may be bulked from pH-variations above and below neutral, entrained carbon dioxide, carbonates, pectin, comminuted organic wastes of light specific gravity, and filamentous organic growths. Automatic pH-control, with lime, aids in producing fresh sludge with a low moisture content. Washing with water, whether from dilute sewage at low flow, storm water, or water added for elutriation, likewise aids.

[1] San. Engr., Bureau of Water Works and Supply, Los Angeles, Calif.

[2] "Sewage Reclamation Plant for Los Angeles," by R. F. Goudey, *Western Construction News*, October 25, 1930, p. 519.

Mechanical mixing in the bottom of clarifiers or by mixing paddles for mechanical flocculation may assist in producing superior fresh sludges.

Bulking of activated sludge may be caused by many physical conditions similar to those for the bulking of fresh sludge. The process of "bulking" is not fundamentally biological because activated sludge can be bulked instantaneously by physical means long before bacteria or fungi could possibly multiply to account for it biologically.[3] Many organic and inorganic compounds that are laxative to man appear to be capable of bulking activated sludge. Pulp of fresh or dried apricots, figs, and tomatoes aerated in water become suspended in a relatively large volume. Onion and similar pulps, first alkalized and then made neutral, will also cause distended and bulky suspensions. When small amounts of activated sludge are added to pure pulp suspensions, the sludge becomes distributed throughout the pulp suspension and appears "bulked." A little activated sludge in a suspension of pure magnesium hydroxide appears enormous. In all these cases the activated sludge is distended and dispersed. Granite dust, sand, lime, or dilution with water, in all these cases, can reduce the "bulk" from 95% to 30% almost instantaneously, just as an increase in the pH-value in the magnesium hydroxide floc causes its immediate concentration. Lime is probably effective also in destroying enzymes, inverting colloidal phases, and in releasing ammonia.

Any sequence of treatment that controls bulking in all types of sludges and removes as large a part, as is practical, of the sewage solids as fresh sludge constitutes the first fundamental step in sludge concentration. The solids in passing preliminary clarifiers into aeration or chemical treatment units are converted into extremely liquid sludges, 97% to 99.5%, which constitute the main problem so far as concentration is concerned.

Excess Activated Sludge.—Three methods of excess activated sludge concentration were investigated by many months of practical and continuous operation. The first procedure was concentration by filling a tank with excess activated sludge, allowing it to settle by batch treatment for 24 hr, withdrawing water from clear zones between sludge and scum, and finally pumping the remaining sludge, which averaged 97.76% water. The second method was the return of excess activated sludge continuously to the preliminary clarifier which raised the moisture in the fresh sludge from 91.96% to 94.42%.

The third method consisted of continuous removal of excess sludge as equivalent aeration tank liquor, settling it at a rate of 600 gal per sq ft per day in a separate clarifier and treating the supernatant liquor to maintain a chlorine dose of 1 ppm over the settled sludge. The chlorine dose was equivalent to 20 lb per million gallons of sewage treated. The chlorine in the waste overflow was used to disinfect the final effluent from the entire plant.

No scum appeared on the thickening tank. Sludge was pumped once daily. This procedure produced an average moisture content of 94%. Washing with water produced sludges as low as 92%. The addition of lime averaged a sludge moisture of 90%. The addition of such inert materials as ground oyster

[3] "Demonstration on Bulking and Debulking of Activated Sludge at Escondido Plant," by R. F. Goudey, *California Sewage Works Journal*, Vol. V, No. 1, 1932–33, p. 61.

shell, decomposed granite, and magnesite produced concentrated activated sludges as low as 76%.

Excess activated sludge in Southern California can be reduced, readily and economically, to a moisture content of 92%, although a limit of 94% is sufficient where the sludge is pumped to digesters. The main secrets are: First, the settling of thin sludge from aeration tanks carrying 1,000 to 1,500 ppm of solids instead of settled sludge from the activated sludge clarifiers; second, settling at the rate of 600 gal per sq ft per day; third, the addition of chlorine of 1 ppm to prevent scum formation and to cause thickening; and, fourth, use of a continuous operating basis. If re-aeration tank liquor or settled activated sludge from the secondary clarifier is used, a smaller concentrating tank is required but the moisture contents will range from 96% to 97%.

In order to visualize the relative value of the foregoing four methods, a tabulation of the resultant combined volumes of fresh and activated sludges for a plant producing one ton of dried fresh solids at a moisture content of 91.96%, and two tons of dried excess activated sludge at 99.8% moisture, is as shown in Table 1.

TABLE 1.—COMBINED VOLUME OF WET, FRESH AND THICKENED ACTIVATED SLUDGE FOR VARIOUS METHODS OF CONCENTRATION

No.	Process	Moisture content (%)	Combined volume (gal)
1	Settled activated sludge mixed with fresh sludge	97.76	24,100
2	Excess sludge pumped to clarifier	94.42	12,900
3	Aeration liquor settled under chlorine and mixed with fresh sludge	92.00	9,000
4	Aeration liquor settled under chlorine and mixed with fresh and activated sludge	90.00	7,200

The City of Phoenix, Ariz., installed a sludge thickening tank[4] for both fresh and excess activated sludge. Dario Travaini, Assoc. M. Am. Soc. C. E., reports that fresh sludge has been reduced to 85% moisture but not without troublesome odors. The fresh sludge has an average moisture content of 95% and the excess activated sludge averages 97.4% moisture. These high values are due to chemical treatment prior to preliminary clarification and also to the fact that the thickening plant is too small to use aeration tank liquor rather than activated sludge itself. Mr. Travaini also reports that the Phoenix plant has "never successfully concentrated activated sludge in the primary clarifier" and that the digesters would "fail completely" if it were not for the sludge concentration tank employing chlorine.

It is apparent that the chlorine settling method is superior to that of pumping excess activated sludge to the preliminary clarifier where good removals are required and where digestion is involved. In case the final effluent needs to be chlorinated, the chlorine serves both for sludge concentration and mixture of the sludge liquor carrying sufficient chlorine to disinfect the final effluent. For moderate removals and where chlorination of the final

[4] "Sludge Thickening at Phoenix, Arizona," by Dario Travaini, *Water Works and Sewerage*, April 1934, p. 107.

effluent is concerned, concentration of activated sludge need not be carried below 94%.

Digested Sludge.—As early as 1930 successful experimental runs were conducted on digested sludge treated with various quantities of ferric chloride and lime followed by vacuum filtration. The average dose was 4% ferric chloride and 10% lime. Digested sludge produced a cake of 65% moisture content at a chemical and filtration cost of $3.50 per million gallons.[5]

Raw Sludge.—It was found that chemical doses calculated on the basis of dry solids used to concentrate digested sludge likewise concentrated fresh sludge; and this procedure permitted equally satisfactory dewatering of fresh sludge for either vacuum filtration or rapid sand bed drying. This indicates that digestion is an unnecessary step where chemical treatment of sludge is justified for other reasons.

Conclusions

All practices that will produce a concentrated sludge—whether fresh, activated, or digested—should be given due consideration. The cheapest way to remove solids of lowest moisture content is in the first clarifier where attention is paid to pH-control of the incoming sewage, mechanical flocculation, timing of sludge removal, and washing and dilution of sludge.

Excess activated sludge can be concentrated best by removing its equivalent as aeration tank liquor, diluted with water to 1,200 ppm and treated with lime occasionally, followed by sedimentation under a 1 ppm chlorine blanket. The withdrawal of aeration tank liquor should be continuous, the settling tank need never be emptied, and the thickened sludge should be pumped not more frequently than once a day. By this procedure, moisture contents of 94% can be attained.

When it comes to producing sludges of maximum concentration, digestion is unnecessary and wasteful. The economy of sludge concentration is based fundamentally on the degree of solids to be removed by a given plant, whether the final effluent needs chlorination, and whether vacuum filtration of sludge is required.

[5] "Digestion and Disposal of Sewage Sludge," by R. F. Goudey, *Western Construction News and Highways Builder*, July 10, 1932, p. 381.

BACK RIVER SEWAGE WORKS, BALTIMORE, MD.

By C. E. Keefer,[6] M. Am. Soc. C. E.

Synopsis

When the preliminary plans for the additions to the Back River sewage works in Baltimore, Md., consisting of activated-sludge units, were made in 1937, it was decided to provide tanks for the thickening of the excess activated sludge. These tanks have been in almost continuous service since they were put in operation on November 27, 1939, and have given satisfactory service.

Description of Thickening Tanks

Two thickening tanks were constructed, each 26 ft in diameter with a working depth of 16 ft. The excess sludge enters the tanks through a vertical central column, and the supernatant flows radially outward and discharges over a V-notched weir at the tank periphery. A mechanism is supported by and rotates about the aforementioned column at a speed of one revolution in 14.75 min. The bottom of the mechanism is provided with squeegees for moving the settled sludge along the tank floor to the sludge discharge pipe near the center of the tank. Vertical steel angles, for thickening the sludge, are also attached to the mechanism. Projecting vertically downward 12 in. below the surface of the supernatant in each tank are 8-in. nozzles through which chlorine water can be discharged.

The two tanks were designed to thicken the excess sludge when the sewage flow treated by the activated-sludge units would amount to 40 mgd. It was estimated that 160,000 gal of excess sludge, containing 1.5% of dry solids, would result from treating the aforementioned quantity of sewage. With this flow of sludge the surface loading, with one thickening tank in service, would be 300 gal per sq ft daily. A total of 13 ft was allowed for deposited sludge and 3 ft for supernatant in the tanks. The supernatant from the tanks is pumped into the settled sewage being treated by the activated-sludge units, and the thickened sludge flows to two plunger pumps. These pumps discharge the material through a 6-in. cast-iron force main into a well, which receives the raw sludge from the preliminary sedimentation tanks. The mixture of raw primary and activated sludges is then pumped into heated digestion tanks.

This treatment of the activated sludge indicates the reasons for thickening it, namely:

1. To reduce pumping costs;
2. To save fuel for heating the sludge; and
3. To reduce the volume of space in digestion tanks occupied by water.

[6] Associate Engr., Bureau of Sewers, Baltimore, Md.

Operating Results

When the thickening tanks were designed, it was expected that each tank would treat the excess sludge resulting from the treatment of 20 mgd of sewage. Operating experience has indicated the desirability of frequently using two tanks even when the sewage flow is less than 20 mgd. Item 5, Table 2, indicates that during the 3-yr operating period from 1940 to 1942, inclusive, two tanks were in service from 59% to 64% of the time. When the sludge index is low one tank is sufficient to thicken the sludge, but when the index begins to increase materially two tanks are necessary.

TABLE 2.—Operating Results and Data, Activated-Sludge Thickening Tanks, Baltimore, Md.

No.	Description	1940	1941	1942
1	Quantity of sewage treated (mgd)	15.61	13.28	11.94
	Depth of Sludge in Tanks (Ft):			
2	From	7.7	4.9	9.1
3	To	14.0	11.7	
	Percentage of Time in Service:			
4	One tank	36	39	41
5	Two tanks	64	61	59
	Sludge to Thickener:			
6	Quantity (gal per million gal of sewage)	46,000	22,520	18,000
7	Solids (%)	0.9	1.1	0.9
8	Volatile matter (percentage dry weight)	77.1	79.5	80.1
	Sludge from Thickener:			
9	Quantity (gal per million gal of sewage)	5,370	4,231	4,500
10	Solids (%)	1.9	2.1	1.4
	Thickener Effluent (ppm):			
11	Five-day biochemical oxygen demand	88	....	....
12	Suspended solids	96	81	197
13	Chlorine applied to unthickened sludge (ppm)	16.5	29.4	10.0
14	Reduction in volume of sludge (%)	88.3	81.3	75.0

The performance of the thickening tanks from 1940 to 1942, inclusive, is indicated by Items 6 to 14 in Table 2. The reduction in the volume of the sludge (Item 14) has varied from 75% to 88.3%, with an increase in solids of from 0.9% or 1.1% (Item 7, Table 2) to 1.4% or 2.1% (Item 10, Table 2). The effluent from the thickeners (Item 11) had an average 5-day B.O.D. of 88 ppm in 1940 and an average yearly suspended solid content (Item 12) varying from 81 to 197 ppm. The chlorine applied to the unthickened sludge varied from 10.0 to 29.4 ppm (Item 13, Table 2).

A marked correlation was observed between the sludge index and the solids, in the unthickened and the thickened sludge (Fig. 1(*a*)). When the sludge index was low, the solid content in the unthickened and the thickened sludge was high; and, when the sludge index was high, the reverse condition was observed. The ratio of the solids in the thickened and the unthickened sludge (R) was also found to vary inversely with the sludge index (Fig. 1(*b*)). This relationship can be expressed by the formula

$$R = 7.94\, s^{-0.3125} \qquad (1)$$

in which s is the sludge index. Eq. 1 is based on three years of operating results, and may require modification when more operating data are available. It is not intended to apply to other than Baltimore conditions. No correlation

was found between the amount of inorganic matter in the sludge and the solid content and the sludge index, perhaps because there was insufficient variation in the amount of inorganic matter in the sludge.

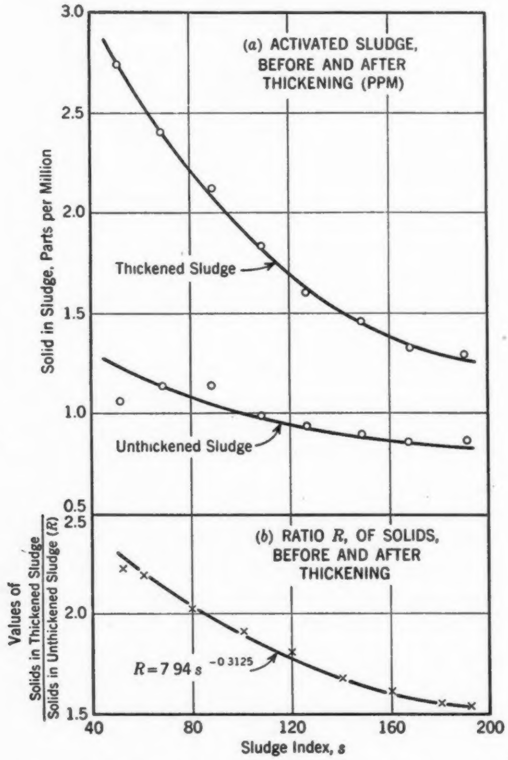

FIG. 1.—RELATIONSHIP BETWEEN THE SLUDGE INDEX AND THE SOLIDS IN THE SLUDGE

CONCLUSIONS

The thickening tanks have amply justified their installation, as they have materially simplified the problem of sludge digestion. The operating results indicate that the degree to which the sludge can be thickened depends upon two things: (1) The solid content of the sludge to be thickened; and (2) the sludge index of the unthickened sludge.

ACKNOWLEDGMENTS

The activated-sludge units were designed by the firm of Whitman, Requardt and Smith, consulting engineers. The Back River sewage works is under the general supervision of Nathan L. Smith, M. Am. Soc. C. E., chief engineer, Department of Public Works, and George E. Finck, M. Am. Soc. C. E., sewerage engineer.

SOME RESULTS IN THE STATE OF NEW JERSEY

By Willem Rudolfs,[7] M. Am. Soc. C. E.

Synopsis

The purpose of sludge concentration or sludge compaction is primarily to reduce the bulk of the sludge and subsequently to effect economy in handling. The method of sludge concentration may be physical, chemical, or both. The ultimate sludge concentration obtained is affected by various factors, chiefly the character of the sludge, its initial concentration, time of compaction, temperature, and pressure. If chemicals are used, the type of chemical is important.

This paper is in the nature of a résumé and deals with several of these factors. The results reported are based upon laboratory and plant scale studies.

The character of the sludge to be concentrated is of greatest general importance. The effects of initial concentration, time of compaction, and temperature during concentration, as well as the effect of the chemicals used, vary for different types of sludges.

The studies on which this paper is based have been made on the three major types of sludges (fresh solids, digested sludges, and waste activated sludges). Because of the mass of data, only some of the results are discussed under separate headings.

Fresh Solids

Initial Concentration.—The reduction in volume of sludge when it is quiescent is naturally greatest with the lowest initial concentration. As an illustration, fresh solids, allowed to compact for 120 hr at a winter temperature of 50° F, with initial concentrations of from 0.5% to 6.2%, show decreasing rates of compaction with increasing initial concentration (Fig. 2). The compaction is greatest during the first 24 hr.

Effect of Time.—Table 3 shows the effect of time on the concentration of fresh solids under actual plant conditions obtained at the Rahway Valley (New Jersey) Sewage Treatment Plant over a period of 3 yr. The average monthly results, irrespective of temperature effects, are shown in Fig. 3. The results confirm closely the laboratory results obtained with fresh solids from the same source as that shown in Fig. 2. Concentration is very rapid during the first 12 hr, followed by a gradual reduction in the rate of compaction.

TABLE 3.—Monthly Average, Fresh Solids Concentration

Hours of concentration	Percentage of Total Solids		
	Minimum	Maximum	Average
12 to 18	3.80	6.90	5.57
24	5.30	8.30	6.42
36 to 48	6.4	8.30	7.19

[7] Chf., Dept. of Water and Sewage Research, New Jersey Agri. Experiment Station, New Brunswick, N. J.

Effect of Temperature.—Experiments and observations made at the Elizabeth (N. J.) Joint Meeting Sewage Treatment Plant, where the fresh solids are concentrated in tanks prior to barging to sea, show that concentration of solids is materially greater in summer than in winter. As an illustration, Fig. 4

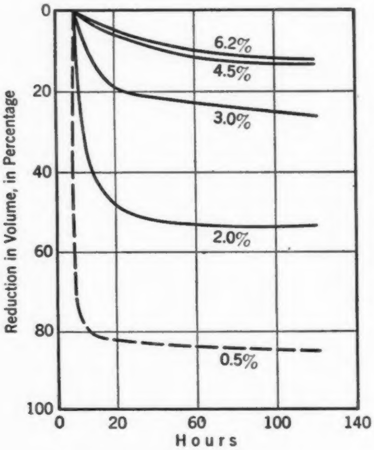

FIG. 2.—RELATION BETWEEN TIME OF COMPACTION AND REDUCTION IN SLUDGE VOLUME

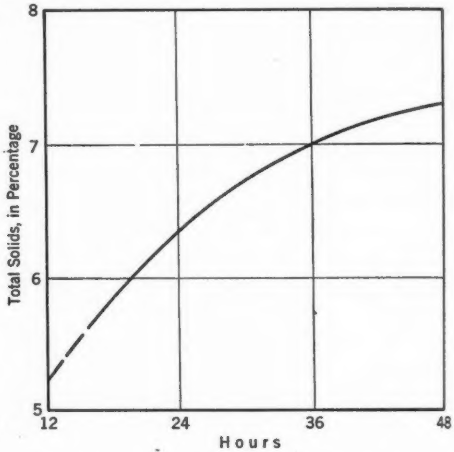

FIG. 3.—AVERAGE RESULTS OBTAINED ON PLANT SCALE DURING THREE YEARS

compares the concentration of fresh solids kept at winter temperatures (47° F to 50° F), with similar solids concentration at summer temperatures (75° F to 80° F). The initial total solids concentration of 5.2% increased during 96 hr to 11.5% in summer as compared with 8.7% in winter. Practice for several years has shown that the sludge volume is materially less in summer than in winter, but handling the less concentrated sludge in winter is more difficult than handling the denser sludge produced in summer. The reason is primarily a question of viscosity of the sludge, which is appreciably greater in winter.[8]

The effect of sludge concentration on cost of operation may be illustrated by the volumes of sludge collected and the percentage reduction in volume obtained during 1940, 1941, and 1942. The sludge pumped in 1940 was 70,700 gal per day, which increased to 74,378 gal per day in 1941. On account of increased solids concentration, however, the reduction in volume amounted to 42.4% in 1941 as compared with 36.1% in 1940, so that less sludge had to be barged in 1941 than in 1940. During 1942 the sewage flow increased, and subsequently the sludge quantity increased to an average daily amount of 91,673 gal, or an increase in volume of 17%. Improved procedures in decanting led to increased solids concentration, reaching an average for the year of 8.14% total solids, so that only 83,000 tons, or about half of the increased quantity of sludge, needed barging.

[8] "Properties of Sludge Which Affect Its Discharge Through 24-In. Pipe," by Willem Rudolfs and Leslie E. West, *Sewage Works Journal*, Vol. XII, 1940, p. 60.

Effect of Temperature on Pumping Time.—The concentrated sludge is barged to sea at approximately two-week intervals. The sludge is pumped through a 24-in. line about 4,000 ft long. The time required for pumping is

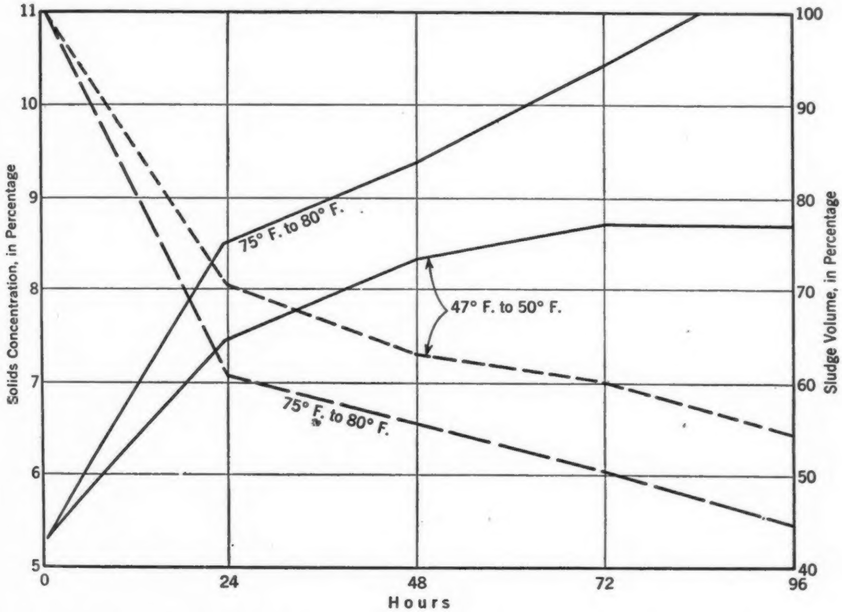

FIG. 4.—SUMMER AND WINTER TEMPERATURES AFFECTING CONCENTRATION OF FRESH SOLIDS

an expensive item in the disposal of the sludge. The temperature of the sludge has a substantial effect on this time (see Table 4). During 1942 the pumping time amounted to 70 min per 1,000 tons of sludge with sludge temper-

TABLE 4.—RELATION BETWEEN TEMPERATURE OF SLUDGE AND PUMPING TIME, 1942
(Elizabeth Joint Meeting Sewage Treatment Plant)

Number of bargings	Temperature range (degrees F)	Total solids (%)	Ash (%)	Tons barged	Average pumping time per 1,000 tons (min)	Average temperature (degrees F)
24	51–73	8.14	25.3	76,070	92	63.5
8	51–57	7.91	22.4	23,870	120	53.5
6	62–66	8.38	28.2	19,145	90	64.5
10	68–73	8.28	29.1	33,055	70	70.7

atures of from 68° F to 73° F, in contrast to 120 min per 1,000 tons of sludge at temperatures of from 51° F to 57° F. Thus, about 80% more time was required during the winter period.

Plotting the 3-yr averages of individual temperatures prevalent during bargings clearly shows the relationship between temperature and time (see

Fig. 5). These results were actually obtained, and the curve cannot be drawn further without data to cover the extension. With the decreasing sludge temperatures the viscosity of the sludge increases, resulting in increased pumping time.[8]

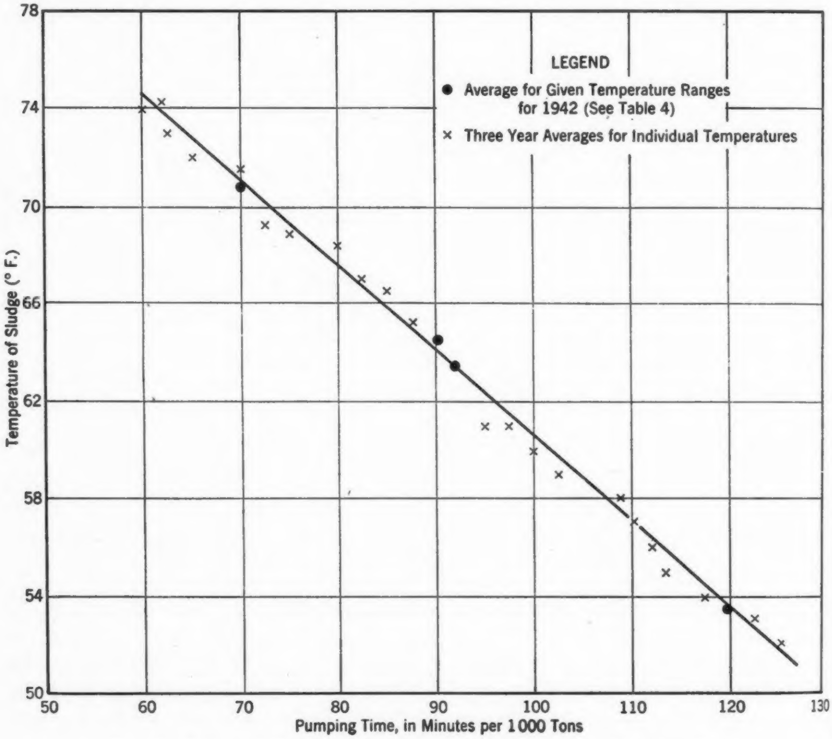

FIG. 5.—RELATION BETWEEN TEMPERATURE OF SLUDGE AND PUMPING TIME OBSERVED UNDER ACTUAL OPERATION

DIGESTED SLUDGE

The primary factors affecting fresh solids concentration—namely, initial concentration, time, and temperature—also affect concentration of digested sludge.

Initial Concentration.—The percentage reduction in volume decreases with increasing initial concentration. Compaction of ripe sludge proceeds in a manner similar to that of fresh solids, but the final solids concentration that can be obtained is greater. However, the fresh solids are more gelatinous in nature than the ripe sludge and they hold water more tenaciously, resulting in less dense sludge.[9]

Effect of Time.—The effect of time on ripe sludge compaction is similar to that on fresh solids. However, with initial higher solids contents, the concen-

[9] "Sludge Compacting," by Willem Rudolfs and Robert P. Logan, *Sewage Works Engineering and Municipal Sanitation*, Vol. 14, 1943, p. 10.

tration of ripe sludge is more rapid than similar concentration of fresh solids with equal elapsed time. For instance, with an initial concentration of 5%, fresh solids may be expected to concentrate 15% in 48 hr on an average, whereas ripe sludge increases under the same conditions an average of 30%. The percentage concentration varies with the length of time allowed for concentration.[10]

Effect of Temperature.—The effect of temperature is less on ripe sludge than on fresh solids. As an illustration, results obtained with ripe sludges of different initial concentrations allowed to compact at summer and winter temperatures are shown in Fig. 6(*a*). The curves indicate that, with an initial concentration of 5% solids, a decrease in volume of about 50% can be expected after 72 hr of compaction. The difference in compaction during summer and

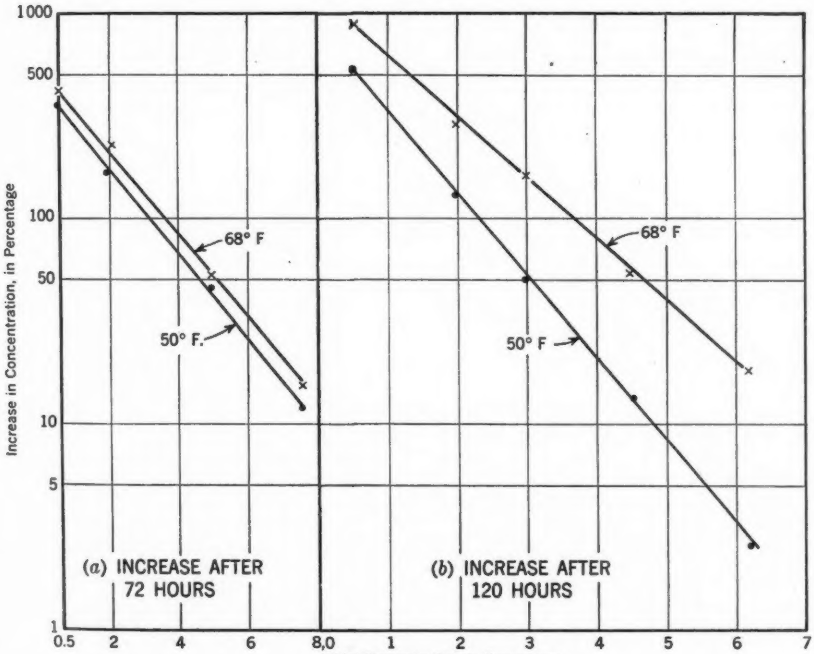

FIG. 6.—EFFECT OF WINTER AND SUMMER TEMPERATURES ON COMPACTION OF DIGESTED SLUDGE

winter is very small as compared with differences in fresh solids concentration (see Fig. 6(*b*)). The greater difference in fresh solids compaction is due mainly to the slimy character of the material and the gases produced during the initial stages of decomposition. The acidic compounds formed during decomposition, together with the gas produced, cause the fresh solids to float, whereas the ripe sludge compacts at the bottom of the container or tank used.

[10] "Effect of Temperature on Sludge Concentration," by Willem Rudolfs and R. P. Logan, *Sewage Works Journal*, Vol. XV, 1943, p. 894.

Activated Sludge

Activated-sludge compaction is subject to the same factors that influence fresh solids and ripe sludge. Material along these lines has been published previously.[11] For that reason, some results obtained with certain chemicals to induce concentration of activated sludge are presented.

Experiments conducted over several years on the compaction of activated sludge produced: Large flocs, small flocs, flotation, dehydration, weighting, retardation of decomposition, and the combination of these methods with the aid of chemicals, inert materials, poisons, and gases.[12]

From the mass of results obtained, some dealing with induced flotation are selected. A flotation process with the aid of chemicals for dewatering of sludge has been used for a number of years. The acid chemicals react with the alkaline substances present in the sludge or liquor, producing numerous small gas bubbles, which in turn cause the sludge particles to rise and aid in the separation of water from the sludge. If, at the same time, such a chemical is found to be a coagulant, which causes the flocs to shrink, prevents septicity, and does not result in materially increased oxygen demand of the liquor, the sanitary engineer is approaching something worth while.

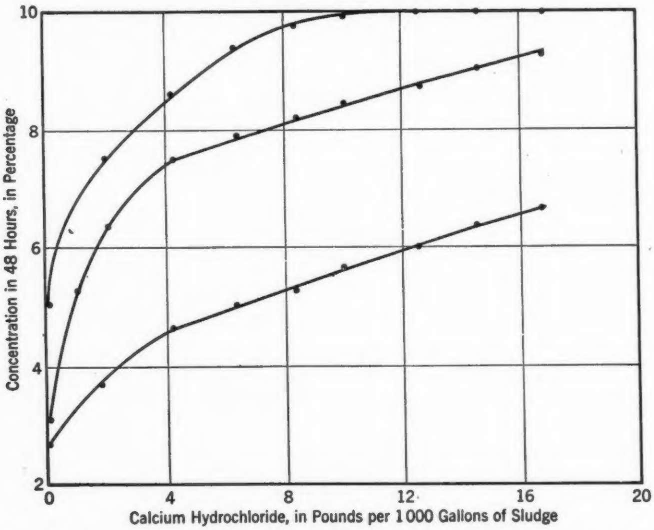

FIG. 7.—CHARACTER OF ACTIVATED SLUDGE IN RESPECT TO THE DEGREE OF OXIDATION

Results obtained with calcium hypochloride indicate that a dense, activated sludge may be obtained by flotation (Fig. 7). The concentration of the original waste, activated sludge (0.30% to 0.60%) does not appear to be a factor, but the character of activated sludge in respect to the degree of oxidation

[11] "Concentration and Distribution of Sewage Solids During Digestion," by Willem Rudolfs and I. O. Lacy, *Sewage Works Journal*, Vol. III, 1931, p. 64.

[12] "Concentration of Activated Sludge by Compacting and Flotation," by Willem Rudolfs, *ibid.*, Vol. XV, 1943, p. 642.

and the size of the floc seems to play a part. The 9% to 10% sludges produced by flotation are leathery in appearance, forming compact, cakelike sludges. The gases produced, which cause flotation, evolve freely when the sludge "cakes" are forming, but the sludge continues to compact materially after the first 24 hr. The chemical need not be added immediately upon discharge of the sludge. Letting the sludge stand for 10 hr at summer temperature and then adding the chemical produced the same sludge density after a total of 48 hr as that produced when the chemical was added at the beginning of the period. Allowing the sludge to become septic likewise had no effect. Neither age nor preliminary compaction, therefore, is detrimental or beneficial.

It is well known that excessive quantities of chlorine added to activated sludge cause a dispersion or have a peptizing action on the sludge. The same reaction occurs with the addition of calcium hypochloride, but to a lesser extent. The biochemical oxygen demand (B.O.D.) of the liquor increases, but the highest value recorded, with quantities of calcium hypochloride three to four times greater than required, was 600 ppm. The liquor can be readily clarified by additions of small quantities of ferric chloride (0.4 lb per 1,000 gal), ferric sulfate, or chlorinated copperas, resulting in a clear liquor with a B.O.D. of about 100 ppm. The iron salt can be added at the same time as the calcium hypochloride. The addition of about 2 lb of sulfuric acid with the calcium hypochloride produces the same result.

The drainability of compacted sludge is enhanced by the calcium hypochloride treatment. The sludge drains readily, forming a thin, dry sheet, without the production of odors on standing. Sludge treated by calcium hypochloride is not improved by vacuum filtration.

Summary

Sludge compaction is affected by the type and character of the sludge, initial concentration, time, temperature, and pressure. If chemicals are used, the type of chemical is important.

Sludges may be concentrated by settling and compaction or by flotation and compaction. Temperature is an important influence on the thickening rate of sludge. The method of compaction to be used depends upon subsequent disposal of the sludge. For barging and incineration, flotation processes appear feasible and may be more economical than compaction by settling.

CONCENTRATION OF SEWAGE SLUDGES, NEW YORK, N. Y.

By Wellington Donaldson,[13] M. Am. Soc. C. E.

Synopsis

Operating experiences and operating data for the sewage treatment plants of New York City are presented in this paper.

Some of the reasons for wishing to concentrate sewage sludges are quite obvious. For instance, in the case of heated digester tanks, it is desirable to introduce the sludge into the digester in well-concentrated condition, to avoid chilling the digesters by unnecessary quantities of cool liquor and at the same time to avoid the necessity of decanting from the digesters an equal quantity of heated supernatant liquor. Likewise, in connection with artificial dewatering and drying processes, it is advantageous to secure initial concentration of sludge. Where the liquid sludge is transported some distance from the plant for disposal, as is the situation locally, concentration is of considerable economic importance.

The density of sludge from primary tanks up to some 10% solids is largely controllable by the operator, depending upon the sludge pumping schedule and method of operating sludge collecting mechanisms. As a matter of fact, difficulties with collector mechanism, pump, or sludge piping from too dense a primary sludge may discourage maximum concentration. Few operators of activated sludge plants will agree that a comparable consistent and dependable control of the density of activated sludge exists. In fact, the difficulty of consistently concentrating activated sludge is the principal demerit to an otherwise very fine process.

In reviewing briefly the experiences of the more important sewage treatment plants of New York City, attention is directed to Table 5, showing the quality of the various sewage sludges.

Coney Island

The Coney Island plant, which, in May, 1942, was increased in capacity from 35 mgd to 70 mgd, employs plain sedimentation for eight months and chemical precipitation and chlorination during the four summer recreational months. All sludge is digested, and the liquid digested sludge is ordinarily transported by scow to natural sand beds or lagoons remote from the plant. There is little significance in the sludge data shown in lines 1 and 2, Table 5, since no particular effort was made to concentrate the raw or digested sludge beyond following a convenient operating schedule.

During the summer months when a ferric salt was used to aid sedimentation, the primary sludge ran about 8.5% solids as compared with the average value

[13] Chf., Bureau of Sewage Disposal Design, Dept. of Public Works, New York, N. Y.

of 7.80% shown in Table 5. Because the digested sludge from this plant is transported by scow to beds or lagoons on Marine Park, it is advantageous to keep digested sludge reasonably dense. However, the 7.71% solids shown for the digested sludge is governed more by the running schedule of the sludge scows than by any manipulation within the plant.

TABLE 5.—Average Quality of Sewage Sludges, New York City

No.	Description	Solids (%)	Volatile (%)
	(a) Coney Island Plant		
1	Primary; 1941	7.80	63.1
2	Digested	7.71	49.0
	(b) Wards Island Plant		
3	Primary; 1941	6.33	74.0
4	Aeration tank effluent	0.176	....
5	Returned sludge	0.48	....
6	Thickened activated sludge to storage	2.28	76.2
7	Weighted average, sludge to storage, primary to excess activated	4.30	....
8	Combined raw sludges to sea	4.45	74.6
	(c) Bowery Bay Plant		
9	Primary; 1941	7.10	68.2
10	Primary; 1942 (eight months)	5.38	62.9
11	Aeration effluent; 1942 (eight months)	0.16	....
12	Returned sludge; 1942 (eight months)	0.40	....
13	Thickened excess activated sludge; 1942 (eight months)	2.40	69.5
14	Combined raw sludges to digesters	4.03	....
	(d) Tallmans Island Plant		
15	Primary; 1941 (January to May)	5.41	64.8
16	Primary; 1941 (June to December)	7.52	56.6
17	Aeration tank effluent (seven months)	0.152	....
18	Returned sludge (seven months)	0.51	....
19	Thickened activated sludge (five months)	1.35	56.5
20	Thickened activated sludge (seven months)	3.48	61.0
21	Combined raw sludges to digesters	6.41	....
22	Digested sludge to storage tank (twelve months)	5.94	....
23	Digested sludge to disposal (twelve months)	7.23	39.4

The wooden scows used for transporting the digested sludge from the plant to Marine Park are owned by the city. Towing is by contract, but the city personnel operate the gasoline driven pump on the scow for discharging the sludge through pipe and hose to the beds. Therefore, unnecessary scow trips would involve additional charges for towing, gasoline, and maintenance. During 1941 the disposal of digested sludge in this manner cost $4.31 per ton of dry solids delivered, capital charges included.

Wards Island

The Wards Island activated-sludge plant treats about 200 mgd of sewage. Both primary and excess activated raw sludge are finally disposed of by barging to sea in special Diesel-propelled sludge vessels, a distance of 34 miles

from the plant. Therefore, it is of considerable importance that the sludge barged to sea be concentrated as much as possible. The primary settling tanks have retention periods of less than 1 hr. This probably limits the concentration of solids pumped from these tanks to the average of 6.33% solids shown in line 3, Table 5. Little can be done to change the density of the primary sludge, but the excess activated sludge requires continued watchfulness to avoid barging unnecessary water to sea. At this plant certain of the final settling tanks are set aside as sludge thickening tanks and operated at a slower rate to give a greater density of sludge than results in the regular final tanks. It will be noted from Table 5 that the return sludge averaged 0.48% solids (suspended solids, 4,800 ppm), whereas the thickened excess sludge from the special tanks averaged 2.28% solids. The computed weighted average of the combined primary and excess activated sludge was 4.30% solids. The difference between this and the succeeding value (4.45) represents a slight concentration secured in the sludge storage tanks by decantation before loading the sludge vessels. Decantation in this manner is practiced sparingly because of the adverse effect of the decanted liquor on the aeration tanks when the liquor is returned to the head of the plant.

Experience in New York City indicates that the rather common practice in smaller plants of concentrating excess activated sludge by wasting return sludge to the head of the primary tanks is not to be recommended unless the primary retention period is 2 hr or more. Shorter retention periods are apt to result in higher suspended solids in the effluent than in the raw sewage influent. Moreover, the septic character of the settled sewage is likely to be disturbing to the subsequent aeration process.

The motor vessels used in transporting sludge to sea are owned and operated by the city with a fixed personnel determined by U. S. maritime regulations. During 1941, each trip (1,500 wet tons) of sludge to sea cost about $202, for operating and maintenance charges. The cost of disposal of sludge to sea by the motor sludge vessels is estimated to be $3.13 per dry ton of solids transported, capital charges included. For New York City, sea disposal is the cheapest of methods available.

Bowery Bay

The 40-mgd activated-sludge plant at Bowery Bay was started on primary treatment in November, 1939, but secondary activated-sludge treatment was not begun until March, 1942. During the early operations the raw primary sludge was barged to sea. Later, as the digesters were put in service, the combined primary and excess activated sludges were digested and the digested sludge in most part was barged to sea by Diesel sludge vessels as at Wards Island. However, substantial amounts of liquid digested sludge have been carted by specially designed motor tank trucks to the unimproved areas of Flushing Meadow Park (formerly the site of the New York World's Fair), there to be used for soil improvement after natural drying.

During the period of primary treatment only, the raw sludge transported to sea averaged 7.10% solids, but the density decreased in the subsequent

year to an average of 5.38%. The Bowery Bay plant has primary tanks of 1-hr retention. Wasting excess activated sludge ahead of the primary tank is not practiced, although a rather heavy decanted liquor from digesters is so wasted, and has proved disturbing to the aeration process. The decreased density in 1942 was probably due to the effect of decanting and to a higher rate of sludge pumping.

For concentration of the excess activated sludge, the plant has two thickening tanks. Provision is made for thickening either the return sludge or the aeration tank effluent—the latter is practiced. The aeration effluent (line 11, Table 5) averaging 0.16% solids (suspended solids, 1,600 ppm) was concentrated in the thickening tanks to 2.40% solids, whereas the return sludge averaged 0.40% (suspended solids, 3,980 ppm). The weighted average of combined primary and excess activated sludge pumped to the digesters was about 4%.

Since the same sludge vessels serve Wards Island, Bowery Bay, and Tallmans Island, the cost of $3.13 per dry ton given for Wards Island therefore applies also to the Bowery Bay plant, although the distance is longer and the loading more time consuming.

The other method of sludge disposal, mentioned previously, may need more explanation. The need for organic soil building material by the extensive park system of New York City led to the purchase of three special motor tank trucks to serve future park areas with liquid digested sludge. The cost of disposal by these trucks is estimated to be $3.50 per ton of dry solids removed, capital charges included, on the assumption that the trucks are kept rolling, which is not the case under wartime conditions.

Tallmans Island

The 40-mgd activated-sludge plant on Tallmans Island was placed in service in 1939. It treats an average flow of 15 mgd. At this plant, as at Bowery Bay, the excess activated sludge is concentrated in two special thickening tanks before pumping to digesters. The thickeners can be fed only with return sludge. During the first five months of 1941, when sludge thickening was under careful study, the thickened excess activated sludge was wasted to the head of the primary tanks having a retention of about 2 hr, and was recovered with the primary sludge. In the second half of 1941, the excess activated sludge, after thickening, was pumped separately to the digesters. When pumping to digesters, an effort was made to obtain the secure maximum concentration.

Table 5(*d*) shows several interesting facts concerning these operations. During the period when excess activated sludge was wasted to the head of the plant, the primary solids averaged 5.41%, compared with 7.52% solids when excess activated sludge was excluded from the primary tanks. With a feed of return sludge averaging 0.51% solids (suspended solids, 5,100 ppm), the thickeners during the first period concentrated the sludge to 1.35% solids, as compared with 3.48% solids during the more exacting second period. Combined sludges to digesters averaged 6.41%.

The difference between the values shown in lines 22 and 23, Table 5, of

"digested sludge to storage tanks" and "digested sludge to disposal" is due to further concentration in the storage tanks by decantation. The digested sludge at Tallmans Island has been partly barged to sea and partly transported by the tank trucks to Flushing Meadow Park.

Special Thickening Tanks

The thickening tanks used at the Tallmans Island and Bowery Bay plants are of the conventional "picket fence" type. The Tallmans Island tank is illustrated in Fig. 8, and the Bowery Bay tank in Figs. 9 and 10. The main

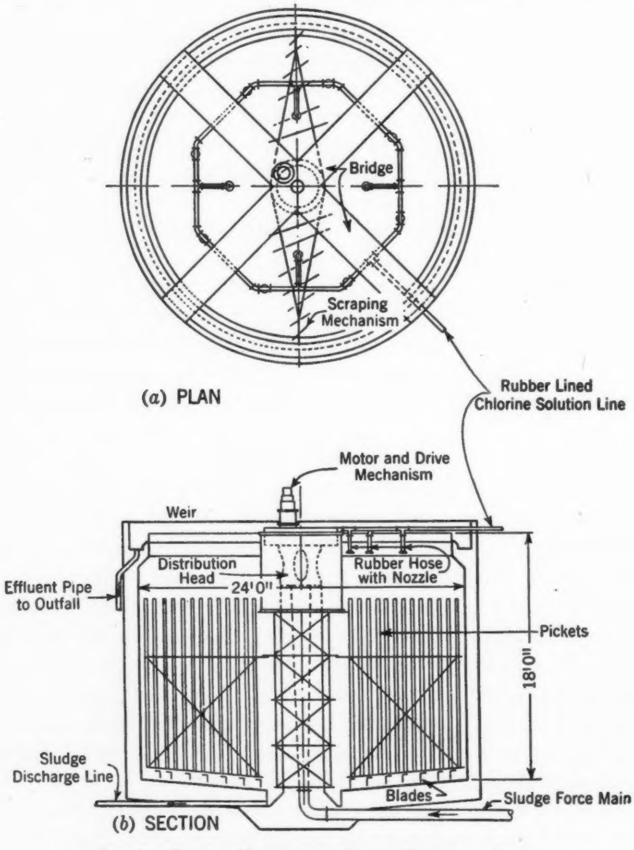

Fig. 8.—Sludge Thickening Tank, Tallmans Island

difference is that the Bowery Bay thickeners, later constructed, are shallower and of greater relative area. The Tallmans Island thickeners were each provided originally with a ring for introducing chlorine solution to the supernatant near the top of the tank, following the practice developed at certain western plants, but it was found that better results were obtained by discontinuing this arrangement and applying the chlorine solution direct to the

incoming sludge liquor. The distributing ring was omitted from the Bowery Bay and later designs.

The use of chlorine in connection with thickening is considered a helpful step during the summer months or on occasions when the settling characteristics of the sludge are not good. In general, chlorine helps when the dissolved

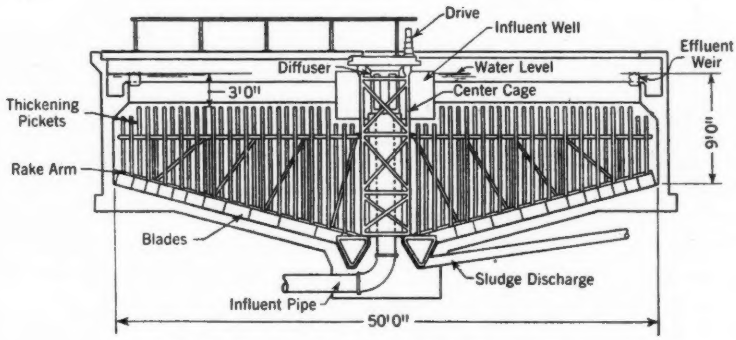

FIG. 9.—SLUDGE THICKENING TANK, BOWERY BAY

oxygen in the entering liquor is low, the sludge index is poor, and the volatiles are high. Experience in New York City shows that better results can be obtained by thickening the aeration tank effluent in a single step than by

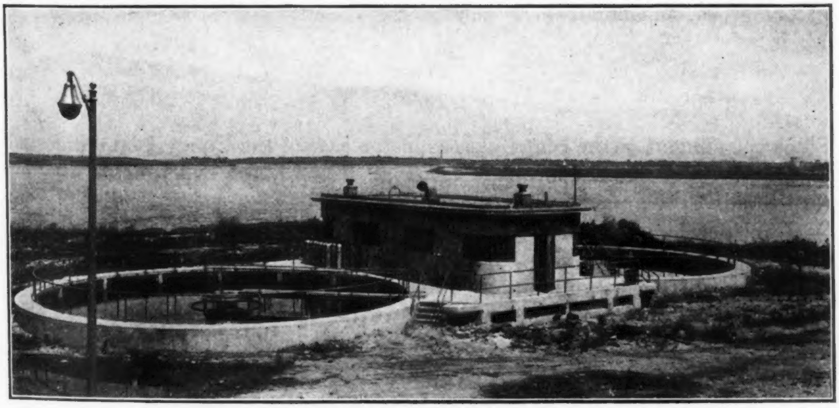

FIG. 10.—BOWERY BAY SLUDGE THICKENING INSTALLATION

rethickening the return sludge. This conclusion seems in agreement with the western experience, although apparently inconsistent with experience at Tallmans Island. It is recognized that the volatile content of activated sludge determines the degree of concentration possible.

At Tallmans Island, an effort to maintain a residual of 1.0 ppm in the overflow resulted in an average dose of 75 ppm of chlorine. At Bowery Bay, under wartime conditions, an average dose of 18 ppm of chlorine was used.

No residual chlorine was maintained in the overflow; hence the treatment could be considered only a palliative for a poor sludge condition and no answer to proper requirements.

Centrifuge Experiments

Aside from the usual operating methods described herein, it may be of interest to note some special experiments with concentrating sludges.

At the Jamaica (N. Y.) plant, a special centrifuge has been installed for the purpose of dewatering material caught by fine screens. During 1938 some thirty-five runs were made with this centrifuge on digested primary sludge from the Coney Island plant. The feed of the centrifuge averaged 11.1% solids; the cake, 50.2% solids; and the effluent, 3.4% solids. It was intriguing to find that within a few minutes digested sludge could be concentrated or partly dewatered to a friable inoffensive material ready to be hauled away and distributed on park areas, but the effluent from the centrifuge was still a sludge containing a high proportion of the grease, and no satisfactory way of disposing of this liquor was discovered.

At Wards Island in 1939, a continuous centrifuge was set up and experimented with to determine its possible utility for concentrating activated sludge. The following values were obtained on return sludge as feed:

Feed	Cake	Effluent
0.398%	4.72%	0.0932%

With excess sludge instead of return sludge as feed, the corresponding figures were:

Feed	Cake	Effluent
1.60%	2.59%	0.95%

Since the effluent of the centrifuge, as in the case of the Coney Island digested sludge, still presented a problem for final disposal, the experiments did not lead to any development along these lines.

Considerable study has been made on activated sludge in the laboratory in an effort to break down its gelatinous character, thus enabling the sludge to be separated from its voluminous water content. It is well known, of course, that ferric chloride of about 3% to 5% of the solids content will aid in this respect, but the cost of such treatment makes it adaptable only where sludge is to be dewatered on vacuum filters.

Some studies have been made of freezing methods suggested by John R. Downes of Plainfield, N. J., for dewatering primary digested sludge.[14] It was found that activated sludge could be concentrated to about 20% solids by freezing, quick thawing, and centrifuging. The liberated water, however, had to be separated immediately after each thawing, or the sludge became septic and less responsive to dewatering. The freezing method did not appear attractive for raw activated sludge.

[14] *Proceedings*, New Jersey Sewage Works Assn., 1939.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

EVAPORATION FROM A FREE WATER SURFACE

By G. H. Hickox,[1] M. Am. Soc. C. E.

Synopsis

Experiments on an evaporation pan, 1 ft in diameter, in still air and under controlled conditions, are described in this paper. An analogy is drawn between heat transfer and mass transfer as they occur in evaporation, and evaporation rates are expressed by means of appropriate dimensionless groups.

The results of the experiments are correlated by the use of these dimensionless groups, and the effects of the variables—vapor-pressure difference, wind velocity, pan diameter, water temperature, air pressure, rim height, color of pan, and depth of pan—are discussed and evaluated.

Suggestions are made for the further study of evaporation records and for obtaining field data.

Introduction

Evaporation records in the United States have been obtained by observations on small pans, the purpose being generally to provide a basis for estimating the evaporation from larger bodies of water such as lakes and reservoirs. Direct measurements on large bodies of water are extremely hard to obtain because of the difficulty of measuring inflow, outflow, and leakage with sufficient accuracy.

It is well known that evaporation from the small pans in use is not the same as the evaporation from lakes and reservoirs and considerable effort has been made to determine the relationship between pan evaporation and lake evaporation. The pans most commonly used are the exposed land pan, the buried land pan, and the floating pan (1).[2] Carl Rohwer, M. Am. Soc. C. E. (1), has assembled data comparing the evaporation from various types of pans and reservoirs, and has computed the coefficients necessary to convert pan evaporation to reservoir evaporation. The coefficients vary with the type of pan, the locality, and the season of the year. The final report (2) of the

Note.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by **March 1, 1945.**

[1] Senior Hydr. Engr., TVA, Hydraulic Laboratory, Norris, Tenn.

[2] Numerals in parentheses, thus: (1), refer to corresponding items in the Bibliography (see Appendix I).

Sub-Committee on Evaporation of the Special Committee on Irrigation Hydraulics of the Society has adopted 0.70 as the value of the coefficient to reduce annual evaporation from the U. S. Weather Bureau Standard Class A evaporation pan to reservoir evaporation. This coefficient is based largely on Mr. Rohwer's data and it is stated that a reasonable range of the coefficient is from 0.60 to 0.82. This represents a variation of 31%.

The variation can be accounted for by taking into consideration the various factors that affect the transfer of heat energy to and from the evaporation pan. The difference between the energy supplied by radiation, and the energy lost by back radiation, conduction, and convection, must be utilized either to heat the water or to evaporate it. If the differences in radiation, conduction, and convection can be established for two pans under similar conditions, it should be possible to compute the difference in evaporation. It should also be possible to apply similar calculations to determine the evaporation from a reservoir.

If such calculations can be made satisfactorily, it will be possible to utilize to much better advantage the large mass of evaporation data that have been collected and summarized (3). Robert Follansbee, M. Am. Soc. C. E. (4), has made an attempt to reduce a great many existing evaporation pan records to evaporation from reservoirs. In doing so, he has used coefficients for each pan which are average annual values and has applied them to monthly and seasonal evaporation. The application of average annual coefficients to monthly records is open to question, particularly since many of the data supporting these annual coefficients were obtained in localities where there is little evaporation in the winter months. An annual coefficient based on such data would not be expected to be quite applicable to Southern California, for example.

Until it has been established that calculations, as outlined, can be made with sufficient accuracy, considerable progress can be made by analyzing the factors that affect evaporation. An understanding of the physical occurrences of the evaporation process would lead to observations of those elements of the problem which would make the transfer of pan records to reservoirs more reliable.

The purpose of this paper is to describe the mechanism of evaporation, to draw an analogy between heat transfer and evaporation, to define the relationship between evaporation and certain basic factors, and to show how the results obtained by a number of different experimenters can be correlated.

How Evaporation Occurs

When water changes its state from a liquid to a vapor it is said to evaporate. The transformation requires energy that is known as the latent heat of evaporation. In order for a given mass of water to evaporate and at the same time to maintain its temperature, energy must be supplied to replace that required for evaporation. If there is no loss of heat across the boundary of the mass other than that due to evaporation, the rate of supply of energy is a measure of the rate of evaporation. This is the basis of one method of calculating evaporation. The actual calculation is difficult because there are several ways

in which energy may be added to and removed from a mass of water, not all of which are susceptible of easy evaluation.

From the kinetic standpoint, evaporation occurs when more molecules pass through the surface from the liquid to the surrounding gas than pass from the gas into the liquid. The molecules composing a liquid are in constant motion. Those near the surface are somewhat restrained in their motion by the fact that they are acted on more strongly by the greater number of molecules below them. Nevertheless, a few of these molecules will have energy enough to break through the surface and escape as vapor. In the space immediately above the liquid, there are molecules of the gas which are also in violent motion. A number of these molecules will strike the surface and occasionally will pass through it into the liquid. The rate at which molecules leave the surface depends on the energy with which these molecules move about in the liquid; that is, on the temperature, or the vapor pressure, of the liquid. Similarly, if the space above the liquid contains only its own vapor, the rate at which molecules enter the liquid depends on the number of molecules present and on the energy, or temperature, of the gas, or its vapor pressure. The net loss of molecules from the liquid, or rate of evaporation, thus depends on the difference between the vapor pressure of the liquid at its temperature and the vapor pressure of the gas above it. The rate of evaporation is somewhat reduced by the presence of the gases above the surface of the evaporating liquid (5).

For evaporation to continue, the molecules leaving the surface must be removed from the immediate vicinity. If they were not so removed, their number would increase to the point where the number of molecules entering the surface would equal the number leaving. The liquid then would be in equilibrium with its own vapor and evaporation would cease. The pressure of the gas immediately above the surface of the liquid would be equal to the vapor pressure of the liquid at that temperature. Ordinarily, the molecules are removed by two processes—diffusion and turbulence. The molecules leaving the surface must first pass by diffusion through a thin layer of gas, known as the boundary layer. The laws governing the diffusion of a gas are well known and if it were possible to measure the vapor concentration on both sides of the boundary layer, as well as the thickness of the layer, the rate of passage of molecules, and hence the evaporation, could be calculated easily. Unfortunately, there is no existing technique for making these measurements. After passing through the boundary layer, the vapor molecules are removed more completely by convection in which turbulence is frequently a governing factor. The laws governing transport of mass by turbulence are fairly well understood, but there are certain fundamental constants entering into the equations which have not yet been sufficiently well established for all conditions of atmospheric turbulence. However, an attempt has been made to calculate evaporation by means of the vapor concentration and wind velocity at two different levels above an evaporating surface with fair results (6).

The process of heat transfer from a solid to a gas is analogous to the removal of vapor particles from an evaporating water surface. A boundary layer of stagnant gas surrounds the solid through which heat is transferred by conduc-

tion alone. Outside the boundary layer, the heat is transferred by convection, frequently aided by turbulence. Heat transfer has been computed successfully by the use of an over-all transfer coefficient, combining the coefficients of conduction and convection, known as a film coefficient. The temperature difference used with this coefficient is the wall temperature of the solid boundary and the temperature of the surrounding gas at a considerable distance. There seems to be no reason why the rate of transfer of vapor particles cannot be calculated in an entirely similar fashion. The potential difference is the difference in vapor concentration, or vapor pressure, between the water-air interface and some far distant location. The evaporation coefficient, of course, must be determined experimentally. Heat-transfer coefficients are correlated by means of dimensionless groups. Similar groups may be used to correlate evaporation coefficients.

Evaporation as a Mass Transfer Process

Evaporation Into Still Air.—According to the kinetic theory, evaporation from a water surface is a function of the difference in water-vapor concentration at the water surface and at a distance, and of the nature of movement of the gas into which evaporation occurs. It is a process of mass transfer which is analogous to heat transfer.

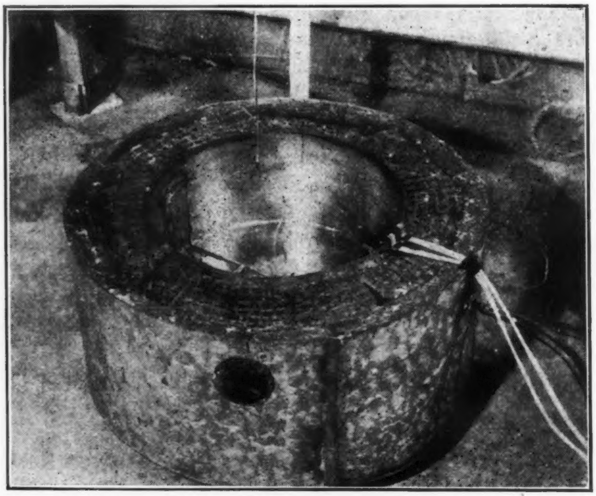

Fig. 1.—Evaporation Pan, Showing Outer Shell, Insulation, Thermocouples for Measuring Temperatures, and Heater Leads. False Floor of Quieting Chamber Removed

Experiments to study evaporation into still air from a circular water surface 1 ft in diameter were undertaken at the University of California at Berkeley. Space for conducting the experiments was made available by the Civil Engineering Department in one of its concrete curing rooms where the temperature and relative humidity were maintained during the course of the experiments

at average values of 71.0° F and 53.1%, respectively. The evaporation pan was made of tin-plated copper, 12 in. in diameter and 6 in. deep. It was insulated by a 4-in. layer of hair felt. Thermocouples were placed at various points within the pan to measure water temperatures. Heat was supplied by an electric heating element immediately below the pan which was enclosed by a quieting chamber 5 ft square and 7.5 ft high in order to reduce stray air currents. A false floor was built and the pan was mounted so that the water surface was flush with it. A few tests were made with the rim of the pan projecting 0.5 in. and 1.5 in. above the false floor. All operations, including filling the pan and measuring evaporation rates and temperatures, were

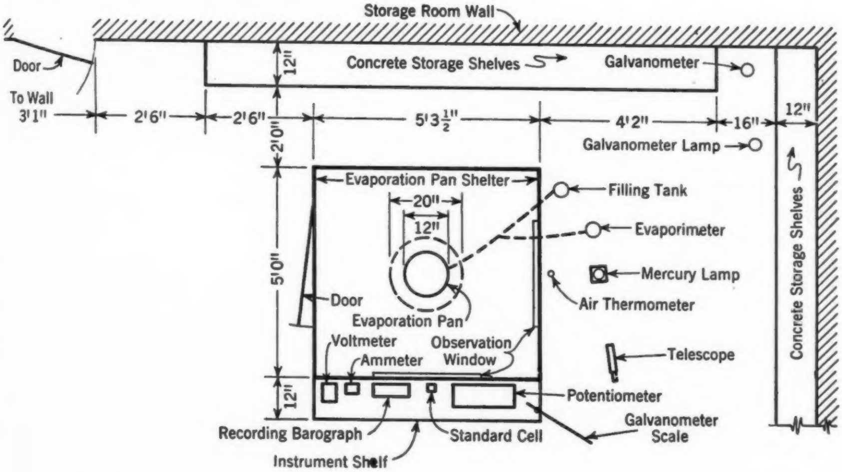

Fig. 2.—Arrangement of Evaporation Pan, Quieting Chamber, and Instruments

conducted from outside the chamber. Water and air temperatures inside the chamber were measured by thermocouples. Air temperature outside the chamber was measured with a thermometer. Relative humidity was measured with a standard sling psychrometer. The heating element was supplied with direct current power, controlled by rheostat, and measured with voltmeter and ammeter. Atmospheric pressure was measured by a recording barograph. Fig. 1 illustrates the evaporation pan (7*b*) and Fig. 2 shows the relative arrangement of apparatus.

The rate of evaporation was measured by an optical interferometer especially designed for the purpose. It consisted of a tilting mirror B and a glass plate A (Fig. 3). Monochromatic light from a source E was reflected toward the plate and mirror, producing interference bands which were then observed through a telescope. The tilting mirror was actuated by a float in a cylinder connected with the evaporation pan. As the water level changed, the number of interference bands visible between two lines etched on the glass plate changed. The change ΔN in the number of bands that occurred in the space a between the lines is a measure of the depth of evaporation E, which can be

expressed by the equation

$$E = \frac{l}{a}\frac{\lambda}{2}\Delta N \qquad (1)$$

In Eq. 1, l = length of lever arm; a = the space between two etched lines on a glass plate; and λ = the wave length of light. All letter symbols used in this paper have been collected and defined in Appendix II. Fig. 3 is a schematic diagram of the evaporimeter and Fig. 4 shows the type of interference pattern produced (7c). This meter was so sensitive that a determination of evaporation rate could be made in from 5 to 15 min. In fact, for high water temperatures the number of bands changed so rapidly that it was difficult to count them.

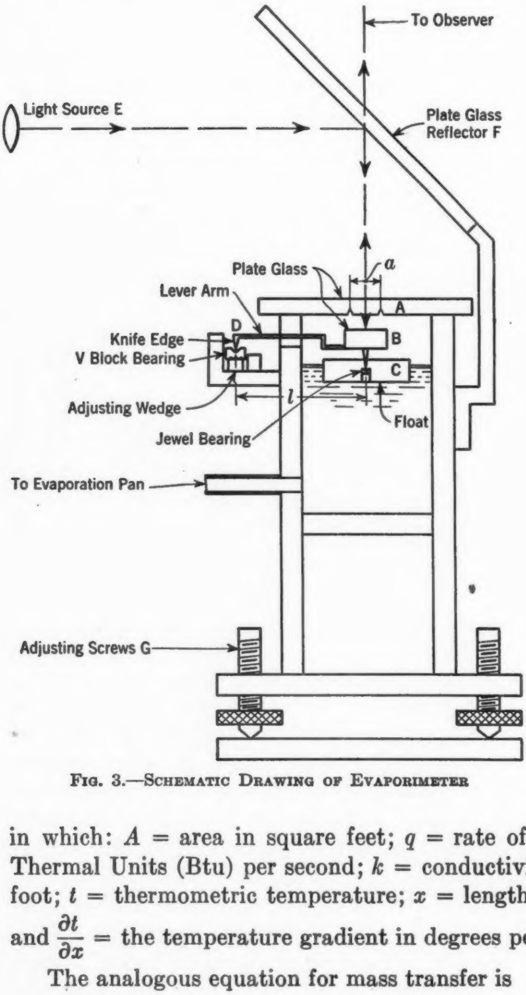

FIG. 3.—SCHEMATIC DRAWING OF EVAPORIMETER

Before making any observations, the temperatures and evaporation rate were allowed to become steady, and were maintained as nearly constant as possible throughout the test.

The basic equation for conduction of heat may be written

$$dq = k\frac{\partial t}{\partial x}\,dA \qquad (2a)$$

in which: A = area in square feet; q = rate of heat transmission in British Thermal Units (Btu) per second; k = conductivity, in Btu per second-degree-foot; t = thermometric temperature; x = length along the path of heat flow; and $\frac{\partial t}{\partial x}$ = the temperature gradient in degrees per foot.

The analogous equation for mass transfer is

$$dw = D_v\frac{\partial c}{\partial x}\,dA \qquad (2b)$$

in which, corresponding to Eq. 2a: w = rate of flow of vapor in pounds per second; D_v = diffusivity in feet^2 per second; c = unit vapor concentration, in

pounds per cubic foot; and $\frac{\partial c}{\partial x}$ = unit concentration gradient in pounds per cubic foot-foot.

Experimental Data.—The results obtained have been plotted in terms of concentration difference and evaporation rate in Fig. 5. Because of the scattering of the points, they have been represented arbitrarily by two straight lines. The equations of these lines are, for $\Delta c < 4.2 \times 10^{-4}$,

$$w_A = 0.0045\,\Delta c \qquad (3a)$$

and, for $\Delta c > 4.2 \times 10^{-4}$,

$$w_A = -\,0.0000059 + 0.0183\,\Delta c \qquad (3b)$$

In Eqs. 3, w_A is the weight rate of evaporation in pounds per (square foot)-second and Δc is the concentration difference in pounds per cubic foot.

When water evaporates from a circular surface into still air so that the vapor is removed by diffusion alone, the rate of evaporation can be calculated with the aid of the following equation developed by H. Gröber and Sigmund Erk (8) for calculating the transmission of heat q through a circular area of radius r into a semi-infinite solid:

$$q = 4\,k\,\Delta t\,r \qquad (4)$$

Applying the analogy between heat transfer and diffusion, the evaporation w_A may be written as

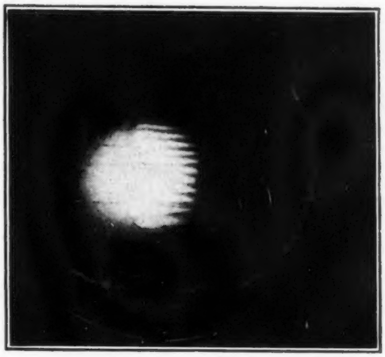

FIG. 4.—TYPICAL INTERFERENCE BANDS PRODUCED BY INTERFEROMETER

$$w_A = \frac{4\,D_v\,\Delta c}{\pi\,r} \qquad (5)$$

This equation is represented in Fig. 5 by the dashed line. The difference between this equation and the experimental values needs explanation. Eq. 5 applies only when the diffusing gas is moving through another stationary gas, which was not the case in these experiments. The movement of water vapor was aided materially by convection currents caused by the heated air next to the water surface, thus increasing the rate of evaporation considerably over that governed by diffusion alone. It is not necessary for the water surface to be warmer than the air, however, in order for convection currents to exist. Since water vapor is lighter than air, convection currents may be set up due to differences in density even when the temperatures of the water and the air at a distance are the same.

For this series of experiments there was a critical temperature of the water surface at which the layer of the air and water vapor immediately above it had the same density as at a distance. When these densities were equal, there should be no convection currents, and evaporation should be governed by diffusion alone. Below this critical temperature, the layer next to the water

surface is cooler than the air above and does not rise. It probably increases in thickness until a slow movement occurs toward the edges of the pan. The evaporation rate for this condition should be slightly greater than that for diffusion due to the movement of the air in removing saturated vapor. Fig. 5

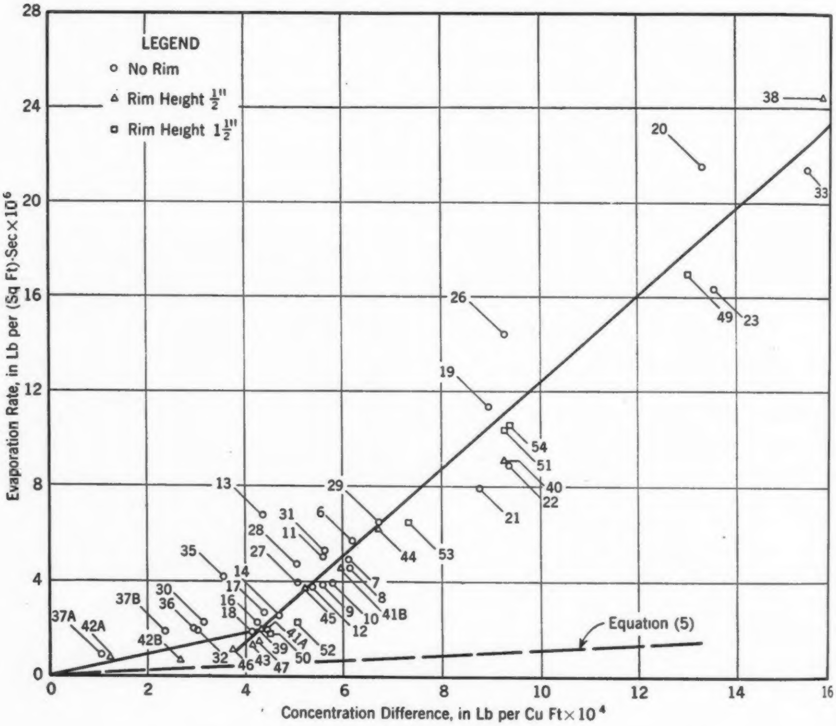

FIG. 5.—EVAPORATION RATE VERSUS CONCENTRATION DIFFERENCE

shows that a break in the curve, with a change in the rate of evaporation, actually occurs at a concentration difference of about 4.2×10^{-4} lb per cu ft. Under the conditions of these tests, the weight of a cubic foot of air and water vapor at a distance was 0.0743 lb per cu ft. Calculation shows that a mixture of saturated water vapor and air has the same unit weight at a temperature of 67.7° F and the unit weight of the water vapor will be 10.7×10^{-4} lb per cu ft. An average of test data for eight runs in the immediate vicinity of the break in the curve of Fig. 5 gave the water temperature as 68.2° F and the unit weight of water vapor at the water surface as 10.77×10^{-4} lb per cu ft. This is excellent agreement.

Examination of Fig. 5 shows that Eq. 5 does not apply even when there seems to be no convection. This is probably due to the fact that it is practically impossible to bring the air absolutely to rest. Under the conditions given, the rate of evaporation given by Eq. 5 is 0.278×10^{-6} lb per (sq ft)-sec, and the

corresponding velocity of the saturated vapor diffusing through air at rest is $V = \dfrac{0.278 \times 10^{-6}}{10.7 \times 10^{-4}} = 0.00026$ ft per sec. It is evident that a slight movement of air over the water surface would increase the removal of water vapor materially.

Dimensionless Representation.—The data of Fig. 5 for vapor transfer can be represented by dimensionless moduli in a manner similar to that for heat transfer by convection. In general, for heat transfer by free convection, experimental data may be represented by

$$N_{Nu} = f\,(N_{Gr}, N_{St}) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6a)$$

in which (see Eqs. 22*a*, 21*a*, and 24*a*, Appendix II): N_{Nu} = Nusselt's number; N_{Gr} = Grashof's number; and N_{St} = Stanton's number. Following the analogy between heat transfer and mass transfer, experimental data on evaporation in free convection may be represented by

$$(N_{Nu})' = f\,[(N_{Gr})', (N_{St})'] \ldots\ldots\ldots\ldots\ldots\ldots (6b)$$

in which (see Eqs. 22*b*, 21*b*, and 24*b*, Appendix II): $(N_{Nu})'$ = Nusselt's number primed; $(N_{Gr})'$ = Grashof's number primed; and $(N_{St})'$ = Stanton's number primed. Eq. 6*b* is commonly written in the form

$$(N_{Nu})' = K' \left[\frac{(N_{Gr})'}{(N_{St})'} \right]^n \ldots\ldots\ldots\ldots\ldots\ldots (6c)$$

in which K' = a coefficient and n = an exponent. Values of Nusselt's number primed, Grashof's number primed, and Stanton's number primed, and

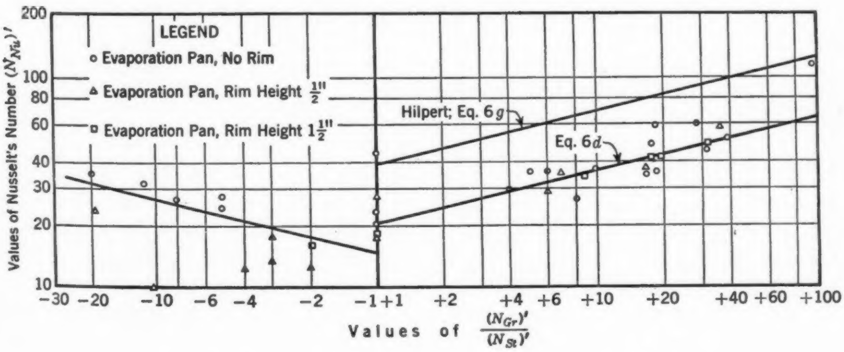

FIG. 6.—NUSSELT'S NUMBER $(N_{Nu})'$ VERSUS THE RATIO $\dfrac{(N_{Gr})'}{(N_{St})'}$

values of the Grashof-prime Stanton-prime ratio were computed from the experimental results for all the tests shown in Fig. 5. Nusselt's number primed has been plotted against the Grashof-prime Stanton-prime ratio in Fig. 6. The best straight line through the points can be represented by the

equation

$$(N_{Nu})' = 0.645 \left[\frac{(N_{Gr})'}{(N_{St})'} \right]^{0.25} \quad \text{(6d)}$$

Here K' and n of Eq. 6c have values of 0.645 and 0.25, respectively.

The only experimental data available for comparison are those of R. Hilpert (9) on moist vertical plates which have been expressed by V. H. Cherry (10) as

$$(N_{Nu})' = 0.94\, (N_{Gr})' f\, (N_{St})' \quad \text{(6e)}$$

If it be assumed that this can be written as

$$(N_{Nu})' = 0.94 \left[\frac{(N_{Gr})'}{(N_{St})'} \right]^{0.25} \quad \text{(6f)}$$

and that $(N_{Nu})'$ should be increased 30% for the upper face of a horizontal surface as suggested by W. J. King (11),

$$(N_{Nu})' = 1.22 \left[\frac{(N_{Gr})'}{(N_{St})'} \right]^{0.25} \quad \text{(6g)}$$

This equation has been plotted in Fig. 6.

Evaporation Into Air in Motion.—The general equations for diffusion of one gas through another gas in motion have not as yet yielded a direct solution to the problem of evaporation into atmosphere. F. Graham Millar (12) has succeeded in approximating actual evaporation records by calculations based on turbulence and eddy diffusivity, but is able to correlate Mr. Rohwer's work only by assuming Mr. Rohwer's anemometer to be in error. However, the diffusion equations have pointed the way to a correlation of experimental results by means of dimensionless moduli. The analogy between mass transfer and heat transfer by free convection has already been shown. A similar analogy for air in motion has been shown by T. K. Sherwood (13).

No experiments were performed to measure the evaporation into moving air, but it is possible to utilize the data of other experimenters. A satisfactory relationship can be expressed by an equation of the form

$$(N_{Nu})' = K\, (N_{Re})^{n} \quad \text{(7a)}$$

in which N_{Re} = Reynolds' number. Eq. 7a can be valid only for values of $(N_{Nu})'$ greater than those for diffusion into still air since, for a given concentration difference, there is a minimum rate of evaporation which is governed by diffusion.

B. C. Shepherd, C. Hadlock, and R. C. Brewer (14) made experiments on evaporation from a free water surface and from saturated sand in pans 1 ft square in a wind tunnel. Water temperatures ranged from 86.4 to 132.8° F, vapor-concentration differences from 0.5×10^{-4} to 16.5×10^{-4} lb per cu ft, and wind velocities from 2.08 to 22.0 ft per sec. Although the data are slightly scattered, they may be represented by the equation

$$(N_{Nu})' = 0.103\, (N_{Re})^{0.75} \quad \text{(7b)}$$

This equation is plotted in Fig. 7.

The inclusion of the data on evaporation from saturated sand is justified because no significant difference from evaporation from a water surface is evident. This is further substantiated by R. W. Powell (15), who performed experiments on evaporation from a free water surface and water retained by linen, filter paper, and gelatin.

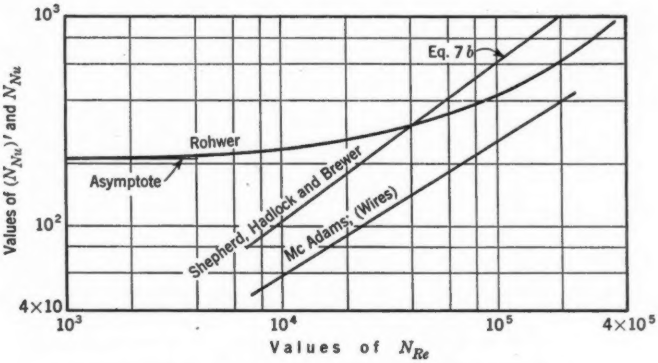

FIG. 7.—NUSSELT'S NUMBERS N_{Nu} AND $(N_{Nu})'$ VERSUS N_{Re}

Mr. Rohwer (16) made experiments on the rate of evaporation E from a 3-ft square pan in a tunnel with controlled wind velocities up to 17.5 ft per sec. He expressed his results in the equation

$$E = C\,(p_S - p_H) \qquad (8)$$

in which C is a constant coefficient and p_S and p_H, respectively, are the pressures of saturated vapor at the temperature of the water surface and the dewpoint. With an assumed average temperature for his experiments, it was possible to compute Nusselt's number primed and Reynolds' number. A curve based on Mr. Rohwer's experiments is plotted in Fig. 7.

These data, although not in sufficient detail to permit an accurate estimate of evaporation, tend to substantiate the analogy between heat and mass transfer. Further experimentation is needed in order to define the curve of $(N_{Nu})'$ versus N_{Re}. At present it may be assumed that $(N_{Nu})'$ varies approximately as the 0.75 power of N_{Re}, with a coefficient of about 0.1 or (compare Eq. 7b):

$$(N_{Nu})' = 0.10\,(N_{Re})^{0.75} \qquad (9)$$

CORRELATION OF RESULTS OF OTHER EXPERIMENTERS

Evaporation Into Still Air.—It has been shown that evaporation into still air probably can be expressed in the form

$$(N_{Nu})' = C\left[\frac{(N_{Gr})'}{(N_{St})'}\right]^{0.25} \qquad (10)$$

which is analogous to the equation used for heat transfer. For atmospheric conditions, $(N_{St})'$ is nearly constant, and may be neglected. Eq. 10 may

then be written

$$\frac{h' L}{D_v} = C\left(\frac{L^3 \gamma \Delta\gamma}{g \mu^3}\right)^{0.25} \qquad (11)$$

in which h' = coefficient of mass (vapor) transfer; L = a characteristic dimension of the surface; γ = unit weight of water vapor and air; g = acceleration of gravity; and μ = coefficient of absolute viscosity. Eq. 11 may be transformed by a number of substitutions and assumptions. The evaporation E is proportional to $h' \Delta c$. Also, $(N_{St})' = \frac{D_v \gamma}{g \mu}$ = constant, and g is constant. It may be assumed, furthermore, that $\Delta\gamma$ is proportional to Δc. Hence

$$E \propto \Delta c \left(\frac{\mu \Delta c}{\gamma^3 L}\right)^{0.25} \qquad (12a)$$

The viscosity varies approximately as $T^{0.50}$ and the unit weight as $\frac{p}{T}$, so that Eq. 12*a* can be written in terms of unit pressure p and temperature,

$$E \propto \Delta c^{1.25} \frac{T^{0.875}}{L^{0.25}\, p^{0.75}} \qquad (12b)$$

The most important variable affecting the rate of evaporation is the difference in vapor concentration. This variable enters to the 1.25 power because of its effect on convection. The experiments on the 1-ft pan described in this paper showed that evaporation into still air varies as the 1.72 power of the concentration difference. J. W. Hinchley (17) found that the exponent was 1.2. J. R. Griffin (18) reports that his experiments combined with those of B. F. Sharpley (7) indicate an exponent of 1.25. Mr. Rohwer's data (16) give an exponent between 1.4 and 2.0. His apparatus was not completely shielded. Also he determined relative humidity at a distance from air which was drawn across the water surface before passing the psychrometer. Undoubtedly, the effect was a reduction in the apparent concentration difference. It also seems probable that the exponent is not constant because of the assumption made in deriving the equation. It was assumed that the difference in vapor concentration was proportional to the difference in unit weight. This would be true if the system were at one temperature throughout. It was stated previously, however, that a vapor-concentration difference could exist without any difference in the unit weights because of the temperature difference. The vapor-concentration difference for which the difference in unit weight is zero depends on both the vapor concentration and the temperature of the air. This may explain the different results obtained by the various experimenters.

According to Eq. 12*b* the evaporation into still air may be expected to increase with the 0.875 power of the air temperature, and inversely as the 0.75 power of the air pressure. There are very few data available, however, on evaporation into still air and it is not possible to check these conclusions.

Most experimenters have developed formulas which express the evaporation in terms of vapor-concentration difference and wind movement in such a way that, for zero wind velocity, the evaporation is directly proportional to the concentration difference. The evaporation rate may be expected to vary inversely as the 0.25 power of the pan diameter. This effect is discussed in the next section. Because of the difficulty of producing absolutely quiet air, and the effect of even slight air movement, as shown previously, the discussion probably has more academic than real value.

Evaporation Into Air in Motion.—Following the analogy between heat transfer and mass transfer by free convection, if it is assumed that evaporation into wind can be expressed by

$$(N_{Nu})' = K\,(N_{Re})^{0.75} \qquad (13)$$

then

$$\frac{h'\,d}{D_v} = K\left(\frac{V\,\rho\,L}{\mu}\right)^{0.75} \qquad (14)$$

Making the same substitutions as before,

$$E \propto \Delta c\left(\frac{V^3\,\mu}{\gamma\,L}\right)^{0.25} \qquad (15a)$$

In terms of temperature and pressure,

$$E \propto \Delta c\,\frac{V^{0.75}\,T^{0.375}}{L^{0.25}\,p^{0.25}} \qquad (15b)$$

It will be noted that the effects of temperature and pressure are different in still and moving air. For high velocities, the exponents given by Eq. 15*b* may be expected to govern, and, for low velocities, they will tend to the value of Eq. 12*b*. It is interesting to examine the results of various experimenters in the light of Eq. 15*b*.

Practically all recent investigators find that evaporation is directly proportional to the difference in vapor concentration at the water surface and at a distance. This is the conclusion reached by Messrs. Rohwer (16), Shepherd, Hadlock, and Brewer (14), G. W. Himus (19), and Millar (12) among recent investigators and by the late D. FitzGerald, Hon. M. Am. Soc. C. E. (20), and J. A. Folse (21) earlier. The later experiments were conducted in wind tunnels with controlled velocities. For any velocity, it was found that evaporation was proportional to the vapor-concentration difference.

Recent investigators who disagree with this conclusion are H. C. Hickman, Assoc. M. Am. Soc. C. E. (22), G. N. Cox, M. Am. Soc. C. E. (23), and C. W. Thornthwaite and B. Holzman (6). Mr. Hickman found evaporation on the Great Lakes for zero vapor-concentration difference. His determination of vapor-pressure difference was based on but one observation of relative humidity per day which he applied to the average of the maximum and minimum air temperatures to obtain the average vapor concentration in the atmosphere. This is entirely in error since the relative humidity varies widely

throughout the day and no single observation can be considered representative. It is quite probable that a vapor-concentration difference existed at the time the evaporation occurred. A more detailed discussion of Mr. Hickman's work, by the writer, has been published elsewhere (24).

Mr. Cox assumed "that the evaporation is a function of the difference between the vapor pressure as determined by the relative humidity and that which would exist for saturated vapor." This conclusion does not seem to be supported by other experimenters or to be justified by the kinetic theory. In fact, for a floating pan, Mr. Cox finds it necessary to introduce the difference of temperature between air and water in order to account for the variable effect of the vapor pressure at the water surface.

Messrs. Thornthwaite and Holzman point out that the movement of water vapor from water surface to air at a distance takes place by means of two entirely different processes—namely, by diffusion through a laminar boundary layer for a short distance and then by turbulence through the overlying atmosphere. They claim that no over-all transfer coefficient can account for the variations that occur in the boundary layer and the turbulent mass because of the varying degrees of roughness which are encountered and which affect both the thickness of the boundary layer and the degree of turbulence. They propose to measure evaporation by means of simultaneous readings of vapor-pressure concentration and wind velocity V at two different elevations H above the laminar layer. The roughness coefficient may be determined from the relationship

$$\log_e Z = \frac{V_2 \log_e H_1 - V_1 \log_e H_2}{V_2 - V_1} \quad (16)$$

Subscripts denote location of observation, 1 being the lower and 2 the upper location. The rate of vapor transfer, which is assumed to be equal to the rate of evaporation or condensation, is given by

$$E = \frac{(K_o)^2 (c_1 - c_2) V_2}{\log_e \frac{H_2}{H_1} \log_e \frac{H_2}{Z}} \quad (17)$$

This method is correct only for an adiabatic atmosphere and is subject to difficulties in the way of determination of constants; but it offers promise as a possible means of determining transpiration and evaporation from soils.

Effect of Wind Velocity.—According to Eq. 15*b*, the evaporation should vary as the 0.75 power of the wind velocity, other things being equal. Mr. Millar (12) found this to be the case for his experiments, and the data of Messrs. Shepherd, Hadlock, and Brewer (14) also conform to this criterion. Mr. Himus (19) found that evaporation varied approximately as the 0.77 power, and Mr. Powell and E. Griffiths' work (25) indicates a variation with the 0.85 power. The conclusions of Mr. Hickman (22) on this point are influenced by his erroneous determination of vapor concentration.

A number of investigators have used wind correction factors of the form $a + b\,V$, in which a and b are constants. A few are listed as follows:

Authority	Factor F
F. H. Bigelow (26)	$1 + 0.07\ V$
D. FitzGerald (20)	$1 + 0.5\ V$
A. F. Meyer (27)	$1 + 0.1\ V$
Carl Rohwer (16)	$1 + 0.268\ V$

Mr. Folse (21) proposed the following correction:

$$F = 0.358\ (V - 10.8) \qquad (18)$$

and R. E. Horton, M. Am. Soc. C. E. (28), gave a curve for velocities as great as 30 miles per hr. All these expressions except that of Mr. Folse are in a form which permits computation of evaporation at zero wind. Mr. Rohwer's experiments, however, indicated that evaporation into still air followed a

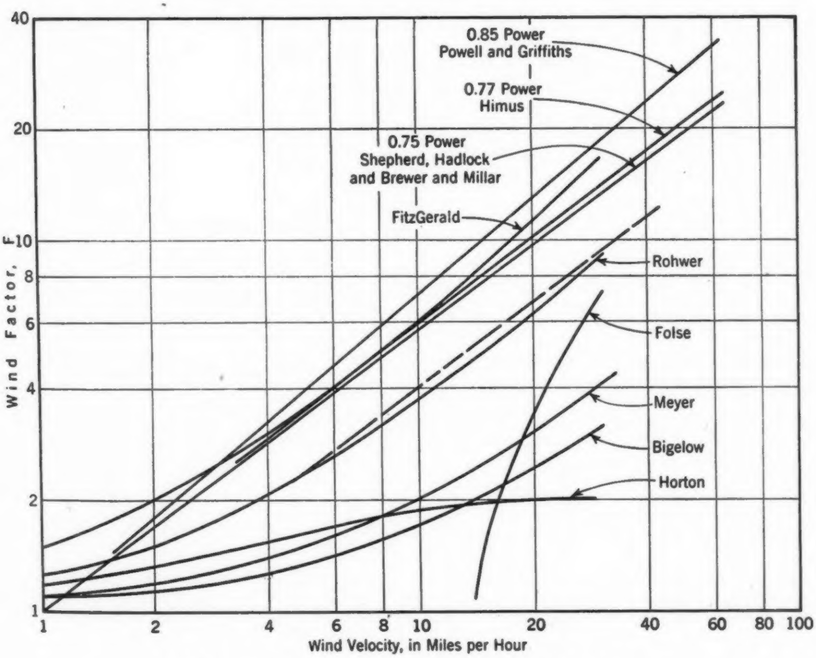

FIG. 8.—RELATION OF EVAPORATION TO WIND VELOCITY

different law. This is in accordance with the present analysis. Furthermore, zero wind does not mean the same for Mr. Rohwer's work as for that of the other investigators. Mr. Rohwer took considerable pains to see that the air surrounding his pan was quiet, whereas the other experiments were made outdoors where zero wind is that movement which is not recorded by an anemometer. The effect of even a slight air movement on evaporation has been discussed previously. For practical purposes, however, absolutely still air seldom exists and the form of the correction is suitable over an appropriate range of wind

velocity. Mr. Rohwer's correction, for example, conforms closely to the 0.75 power law between 5 and 30 miles per hr, whereas Mr. Bigelow's curve has the same slope between about 20 and 100 miles per hr. The exponential law has better theoretical and experimental support for velocities above about 2 miles per hr but cannot be used for extrapolation to zero wind. It can be very useful, however, in correlating results and in moderate extrapolation. The various correction factors are plotted in Fig. 8. The exponential forms have arbitrarily been given a value of unity at 1 mile per hr. For the purposes of this paper, the dashed line of Fig. 8 has been adopted as the best available wind correction factor.

Effect of Pan Diameter.—The effect of pan size has been studied by a number of investigators and the results are summarized herein.

TABLE 1.—EFFECT OF PAN SIZE ON EVAPORATION

No.	Description	Square pan	Circular pan	Reservoir
	Dimension	3 ft by 3 ft	Diameter, 4 ft	1,800 acres
1	Difference in vapor pressure (inches of mercury)	0.640	0.639	0.672
2	Wind velocity (miles per hour)	1.94	1.90	1.68
	Relative Evaporation:			
3	Observed	1.33	1.28	1.00
4	Corrected	1.32	1.29	1.00

Mr. Rohwer (1) compared evaporation from a 3-ft square pan and a 4-ft circular pan with that from a 1,800-acre reservoir with the results given in lines 1, 2, and 3, Table 1. Correcting the evaporation from the pans to the

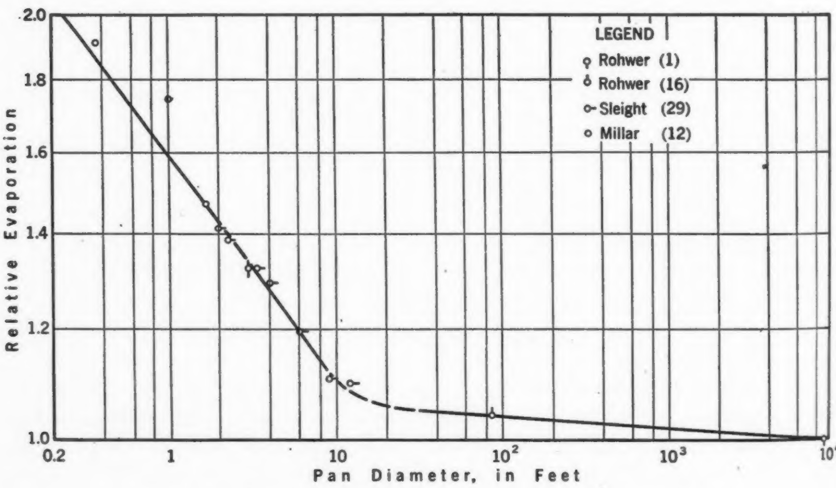

FIG. 9.—RELATION OF EVAPORATION TO DIAMETER OF SURFACE

same vapor-pressure difference and wind velocity as that for the reservoir on the assumption that the evaporation varies directly with the vapor-pressure difference and with the wind as shown by the dashed curve in Fig. 8, the

relative evaporation rates are as shown in line 4, Table 1. The shape of the reservoir is not known, but if it is circular, its diameter is approximately 9,000 ft. These points are plotted in Fig. 9.

Mr. Rohwer (16) also compared evaporation from three different types of pans with that from an 85-ft reservoir. Only one of the pans was comparable, however, the others having different exposure. He found that evaporation from a 3-ft square pan was 26.6% greater than from the reservoir. Assigning the previous value of 1.32 to the 3-ft pan, the relative evaporation from the 85-ft reservoir is 1.043. This value is plotted in Fig. 9.

The late R. B. Sleight, Assoc. M. Am. Soc. C. E. (29), gave the relative rates of evaporation from circular pans of different diameters as indicated in Table 2. The last line of Table 2 gives the rate of evaporation relative to the

TABLE 2.—Relative Rates of Evaporation Showing Effect of Pan Size on Evaporation

Description	Diameter of Pan, in Feet							
	12	9	6	4	3.39	2.26	2	1
Compiled by R. B. Sleight.	1.00	1.009	1.089	1.175	1.202	1.260	1.284	1.589
Compared with 1,800-acre reservoir as unity.	1.099	1.108	1.196	1.290	1.320	1.383	1.410	1.745

1,800-acre reservoir, assuming that the 4-ft pans of Messrs. Sleight and Rohwer were comparable. These data are also plotted in Fig. 9.

Mr. Millar (12) found that the evaporation from a 50.8-cm pan was only 0.77 as great as from an 11.2-cm pan. Fig. 9 shows that the evaporation from a 50.8-cm (1.67-ft) pan relative to a large reservoir is about 1.47. The corresponding evaporation from the 11.2-cm (0.37-ft) pan is 1.91. This point is also plotted in Fig. 9. Two straight lines have been drawn through the points to represent the effect of pan size on evaporation. For pans smaller than, say, 10 ft, the evaporation varies as the -0.16 power of the diameter, and for areas larger than 50 ft in diameter about as the -0.01 power. It is doubtful if the transition occurs abruptly; it may be more gradual, as indicated by the dashed line.

The results are not in excellent agreement with Eq. 15*b*, but are close enough to indicate that the derivation of the equation may be valid.

Effect of Water Temperature.—According to Eq. 15*b*, the rate of evaporation should vary approximately with the 0.375 power of the absolute water temperature. The effect is difficult to find as a variation of 30° at a temperature of 70° F indicates a change in evaporation rate of only 2.1%. This is probably negligible in view of the other uncertainties.

Effect of Air Pressure.—A number of investigators have included a pressure term in their expression for evaporation. T. Russell (30) decided that evaporation varied inversely as the pressure, and Mr. Millar (12) decided it varied as the -0.25 power. On the basis of scanty data, Mr. Sleight (29) gave values which indicated that the evaporation varied inversely as the square of

the pressure. Mr. Rohwer corrected the evaporation computed for Fort Collins, Colo., to other elevations by use of the factor

$$G = 1.465 - 0.0186\,B \qquad (19)$$

in which B is the barometric pressure in inches of mercury. Mr. Millar found that Mr. Rohwer's data were consistent with the -0.25 power of the pressure.

TABLE 3.—EFFECT OF PRESSURE ON EVAPORATION

No.	Location (1)	Elevation (2)	Barometric pressure[a] (3)	Wind velocity[a] (4)	Air temperature[a] (5)	Daily evaporation[a] (6)	Vapor pressure[a] (7)	Wind factor $f(V)$ (8)	$f(B)$ (9)
1	Imperial, Calif......	−68	29.86	1.93	81.7	0.388	0.641	1.52	0.398
2	Fort Calhoun, Nebr.	1,160	28.77	1.35	67.1	0.175	0.300	1.36	0.428
3	Logan, Utah.......	4,778	25.07	1.89	68.0	0.291	0.470	1.50	0.412
4	Fort Collins, Colo...	5,000	25.03	1.01	55.0	0.149	0.268	1.27	0.430
5	Lake Tahoe, Calif...	6,300	23.76	1.02	55.1	0.222	0.366	1.27	0.478
6	Victor, Colo........	10,089	20.43[b]	3.06	50.5	0.161	0.186	1.81	0.478
7	Pikes Peak, Colo....	14,109	18.08	3.98	40.1	0.108	0.112	2.06	0.468
			18.04	10.9	41.4	0.202	0.123	4.23	0.388

[a] All mean values, with units as follows: Col. 3, inches of mercury; Col. 4, miles per hour; Col. 5, degrees Fahrenheit; Col. 6, inches; and Col. 7, vapor-pressure differences in inches of mercury. [b] Estimated from altitude.

This may be confirmed by a brief analysis of Mr. Rohwer's data on the basis of the previous discussion. His results are summarized in Cols. 3 to 7, Table 3. In accordance with the conclusions drawn regarding the variation of evaporation with vapor-concentration (vapor-pressure) difference, wind

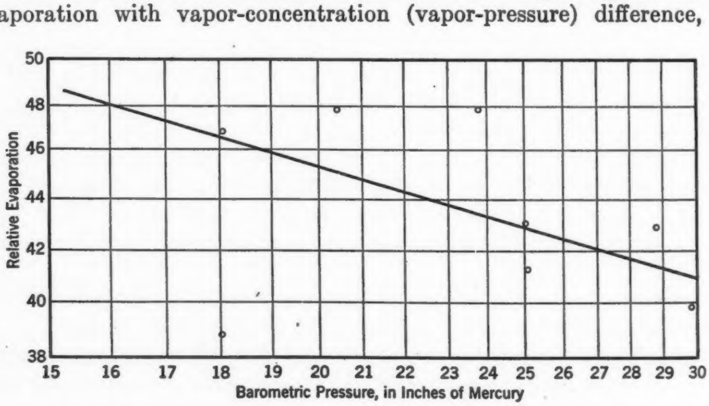

FIG. 10.—RELATION OF EVAPORATION TO BAROMETRIC PRESSURE

velocity, and temperature (neglected), the rate of evaporation can be expressed as

$$E \propto \Delta c\, f(V)\, f(B) \qquad (20)$$

in which $f(V)$ is a wind factor taken from Fig. 8. Values of $f(V)$ are given in Col. 8, Table 3.

Values of $f(B)$ obtained by dividing the observed evaporation by the vapor-pressure difference and the wind factor are given in Col. 9, Table 3. They are plotted against the pressure in Fig. 10. The line on this plate has been drawn arbitrarily with a slope of -0.25 and seems to represent the data as well as any other line that could be drawn through the points.

Eq. 15*b* is seen to be justified for representation of the effects on evaporation of vapor-pressure concentration difference, temperature, size of pan, air pressure, and wind velocities greater than about 5 miles per hr. It may be used for the correlation of experimental results and for moderate extrapolation.

Effect of Rim Height.—Three other variables have been observed to have an effect on evaporation from a pan although they have not been included in Eq. 15*b*. These are the height of rim above the water surface and the color and depth of the pan. R. W. Revell, Jun. Am. C. E. (31), attempted to evaluate the effect of rim height on evaporation from a Class A pan. He

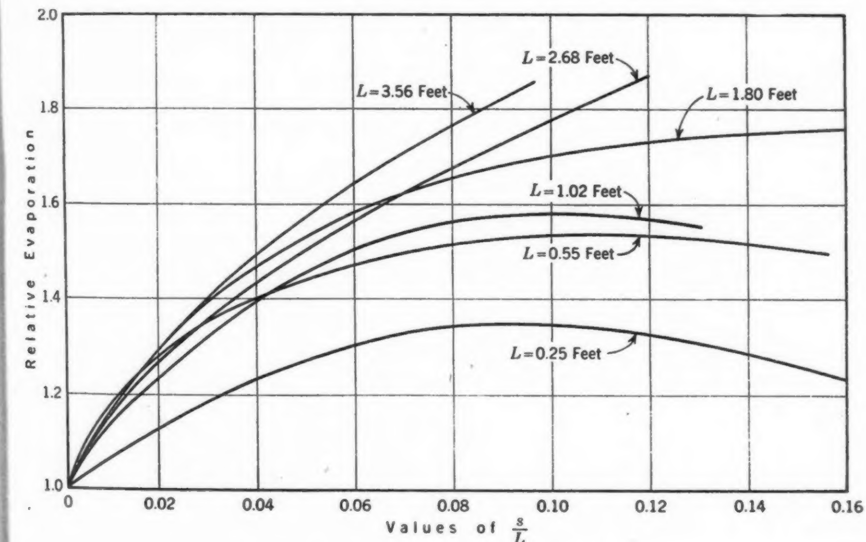

FIG. 11.—EFFECT OF RIM HEIGHT ON EVAPORATION

made experiments on pans having rim heights of 2 in. and 3 in. and found that more evaporation occurred into a wind from the pan with the 3-in. rim height during periods of moderate to high winds, but that less evaporation occurred from this pan during periods of low wind velocity. He suggested that the explanation of this difference in behavior lay in the difference in the size of the turbulent eddies caused by the passage of air over the rim of the pan. His data are so scattered that it is difficult to determine the effect quantitatively. He noted a higher water temperature in the pan with the 3-in. rim height, however, and attributed it in part to the lesser mass of water in the shallower pan. This assumption seems to be reasonable and is consistent with greater evaporation from this pan.

Mr. Powell (15) measured the evaporation from a cylindrical saturated surface in an air stream parallel to its axis. He found that the rate of evaporation was increased when a collar was placed around the cylinder at its leading end. He attributed the increase in evaporation to the increased turbulence of the air stream caused by the projecting collar. In Fig. 11, prepared from his data, the relative evaporation rate has been plotted against the ratio of height of collar to length of evaporating surface. It may be seen that the data for lengths greater than 2 ft and ratios of s/L less than about 0.08 may be represented with fair accuracy by a single curve. This is the range in which the Weather Bureau Standard Class A pan operates. The data by Mr. Powell are probably not strictly applicable to flat surfaces but the curve indicates that, if the rim height of a standard 4-ft pan be increased from 2 in. to 3 in. by evaporation or other loss, the relative evaporation rate in a wind of 6.7 miles per hr increases from 1.47 to 1.61, an increase of about 10%.

Effect of Pan Color.—The color of an evaporation pan has been observed to have an effect on the evaporation rate. This is undoubtedly due to the difference in absorption of solar and sky radiation which affects the water temperature and consequently the evaporation. A series of observations was made by the writer for one year, at Berkeley, on four pans of identical size, shape, and exposure. The pans were of galvanized steel, 1 ft square and

TABLE 4.—EFFECTS OF PAN COLOR AND DEPTH ON RATE OF EVAPORATION

(Inches per Month)

Month	(a) COLOR OF PAN:				(b) DEPTH OF PAN (FEET):			
	White	Gray	Galvanized	Black	0.25	0.50	1.00	2.00
1933								
May[a]	3.42	3.79	3.72	4.14	4.06	3.72	3.74	3.92
June	7.20	8.10	8.12	8.52	8.43	8.12	8.26	8.43
July	8.69	9.13	8.94	9.85	9.14	8.94	9.07	9.68
August	7.27	7.41	7.57	7.89	7.42	7.57	7.85	8.38
September	5.77	5.90	6.09	5.94	6.01	6.09	6.31	7.04
Sub-total	....	....	....	....	35.06	34.44	35.23	37.45
October	4.09	4.08	4.12	4.04	3.36	4.12	4.33	4.52
November	2.19	2.08	2.29	2.21	1.95	2.29	2.62	3.10
December[b]	0.69	0.61	0.59	0.68	0.52	0.59	0.73	0.85
1934								
January	1.22	1.25	1.25	1.29	1.06	1.25	1.34	1.54
February	1.31	1.21	1.07	1.35	1.42	1.07	1.25	1.27
March	2.50	2.79	2.46	3.00	2.74	2.46	2.50	2.65
April	5.28	5.51	5.50	5.96	5.26	5.50	5.47	5.74
Total	49.63	51.86	51.72	54.87	51.37	51.72	53.47	57.12

[a] Sixteen days only. [b] Twenty-three days only.

0.5 ft deep, and were buried in the ground so that the upper edge was just above the ground surface. They differed only in their color. Three were painted white, gray, and black, respectively, using lead base paint, and the fourth was left unpainted. Table 4(*a*) shows the results obtained. The

black pan which should have had the greatest absorptivity showed 10.8% more evaporation for the year than the white pan and 5.8% more than the gray pan. The gray and galvanized pans showed almost the same results. The differences were not constant throughout the year, however, and, in some months, the annual trend was reversed. For example, from November to February, the evaporation from the white pan was greater than that from the gray pan. Significantly, the greatest differences were found during the months of maximum solar radiation. Similar effects have been noted by other investigators. Mr. Cox (23) found that, when a galvanized pan was painted with an asphaltic paint, the evaporation rate was increased about 10%. Mr. Rohwer (16) found a difference of 4% between two different brands of asphaltic paint.

Effect of Pan Depth.—The depth of an evaporation pan also has an effect on the evaporation rate. Very few data are available for exposed land pans but a few comparisons have been made for buried pans. The writer measured evaporation from four galvanized pans 1 ft square with depths of 3, 6, 12, and 24 in., respectively. The data summarized in Table 4(*b*) indicate that evaporation increased as the pan depth increased. This is not borne out by Mr. Sleight's results (29), which are as follows:

Depth of pan (ft)	Relative evaporation
0.25	127.9
0.75	126.8
1.25	127.9
1.75	127.9
2.75	128.4
5.75	127.4

The lack of agreement may be explained in part by the fact that Mr. Sleight's observations, with the exception of those on the 2.75-ft pan, were made during the period, May to September, whereas the Berkeley tests extended throughout the year.

Inspection of Table 4(*b*) shows that the relative evaporation for the Berkeley tests varies with the season. The totals for the shorter period are more nearly in agreement with Mr. Sleight's results in that the evaporation from the 3-in. pan is seen to be greater than that from either the 6-in. or the 9-in. pan. The temperature of the deeper pans appeared to be less affected by radiation than that of the shallowest pan because their masses were proportionately greater relative to the surface which received and emitted radiant energy. This led to higher temperature in summer and lower in winter, with a corresponding effect on the relative evaporation rates. Also, the deeper pans remained at a more nearly constant temperature because of the stabilizing effect of the earth with which they were in contact.

Correlation by Other Investigators.—Other investigators have also attempted to correlate their own results with those of others. Mr. Millar (12) was able to explain Mr. Rohwer's results with a fair degree of satisfaction, although he

found what seemed to be a discrepancy in the wind velocities observed by Mr. Rohwer.

Mr. Powell (15) plotted $\frac{E\,L}{p_S - p_H}$ against $V\,L$ and was able to obtain a fair correlation of the results reported by Messrs. Millar (12), S. K. Banerji and H. M. Wadia (32), FitzGerald (20), T. B. Hine (33), Rohwer (16), H. Thiesenhusen (34), M. Lurie and N. Michailoff (35), and Shepherd, Hadlock, and Brewer (14). This excellent contribution should be read by all who are interested in this phase of the subject.

The Energy Balance Method of Calculating Evaporation.—Evaporation has been computed by several investigators by means of an energy balance. In this method, the total energy input (solar radiation), less the energy losses (back radiation, conduction, convection, increase of water temperature, and evaporation) must equal zero. If all quantities are known except the evaporation, this is determinable as a residual. The method has been described by Wilhelm Schmidt (36) and A. Angstrom (37) and later by N. W. Cummings (38)(39), Burt Richardson (39)(40), and others. These writers applied the method to lakes and oceans so that conduction through the walls was not a problem as with a pan. Messrs. Richardson and Cummings also used insulated pans.

In all cases the heat lost by convection has been an uncertain quantity. It was assumed by Mr. Schmidt that there was a ratio between the heat lost by convection and that used in evaporation since the processes of heat transfer by convection are analogous. I. S. Bowen (41) made the same assumption later and wrote at length to substantiate it, although in the meantime Mr. Angstrom (37) had shown the impossibility of such a ratio, pointing out that, if it existed, there would be, for a temperature gradient of zero, either no convection and no evaporation, or convection and a corresponding evaporation. It is obvious that both conditions are absurd. The first condition might be satisfied if the atmosphere were saturated with water vapor, but the second condition is impossible. Furthermore, Mr. Bowen assumed equilibrium conditions which seldom exist. Water temperatures are constantly rising and falling, so that the storage of heat is a considerable item in the energy balance, at times amounting to many times the energy used for evaporation. Under these conditions, the ratio varies considerably and, in fact, loses its significance entirely. Mr. Bowen's ratio can be significant only if applied to average conditions over a period of time long enough so that storage of heat is not a factor.

Evaporation Measurement in the Field

Instrumental Observation.—The factors entering Eq. 15*b* are the evaporation rate, the vapor-pressure difference, the wind velocity, the air temperature, the pan diameter, and the barometric pressure. The determination of these factors and their relationships requires field measurements of water surface temperature, air temperature, relative humidity, wind velocity, pan diameter, and barometric pressure. The methods of making these measurements depend somewhat on the purpose for which they are desired. Daily averages may be

suitable for comparing evaporation at two stations which have similar climatic conditions, but are valueless for purposes of careful investigation. The distribution and range of temperature and humidity, for example, throughout the day, have a marked effect on the total evaporation. Most of the day's evaporation occurs when the water temperature is higher than the average, and the vapor pressure at the water surface is high. If the temperature were uniform throughout the day, the higher vapor pressure causing evaporation would not exist. Hence the average temperature has no particular significance with respect to daily evaporation, unless the range of temperature is also specified. Similar considerations apply to the measurement of air temperature and relative humidity since they govern the vapor pressure of the atmosphere. Probably much of the difficulty that is experienced in correlating field observations of evaporation lies in the indiscriminate use of daily averages for temperature and humidity.

A suitable method of measuring the evaporation rate depends on the purpose for which the information is desired. A careful study of evaporation requires that its rate be determined over short periods while the accompanying conditions are sensibly constant. For field data, daily totals will often be sufficient. The optical interferometer described in this paper is sensitive to half the wave length of the light used (about 0.00001 in. for green light), and is quite satisfactory where evaporation rates are desired over very short periods and where the instrument can be set on a very rigid support. It is suitable for experimentation under controlled conditions. Mr. Rohwer (1) described an optical lever arrangement that gave good results in the laboratory. The Tennessee Valley Authority secures a continuous record of evaporation from a standard Class A pan by mounting it on a balance with an attached device that prints the total weight of platform, pan, and water every 15 min. The standard hook gage is quite suitable for measuring daily totals, but is not accurate enough for short-period measurements.

The most important variable in Eq. 15*b*, after the evaporation rate, is the vapor-concentration difference. Its evaluation requires measurement of water temperature, air temperature, and relative humidity. The mechanics of the kinetic theory indicates that the water temperature should be measured as close to the surface as possible. The vapor concentration immediately above the water surface is dependent on the activity (temperature) of the molecules which pass through the surface. The temperature of a layer of water having a thickness equal to the mean free length of path of the molecules next to the surface is the temperature which governs the vapor concentration immediately above the water surface, and the rate of evaporation. This layer is very thin, probably only a fraction of a millimeter. Normally the water temperature increases from sunrise until about 2 p.m. by reason of solar radiation. During this period, the water temperature is highest at the surface. This is explained by Fig. 12, which shows that about 18% of the solar energy incident on the water surface is absorbed in the first millimeter of depth, causing a localized heating at the surface. The surface water, being warmer, does not sink, but remains on top. Typical measurements made in the middle vertical of a standard Class A evaporation pan with a mercury thermometer showed a

temperature of 69° F with the bulb just under the surface, 67° one inch down, 65½° at mid-depth, and 64½° at the bottom. Since the bulb of the thermometer was about 4 mm in diameter, the temperature in the top millimeter was undoubtedly still higher. A small thermocouple placed just under the surface

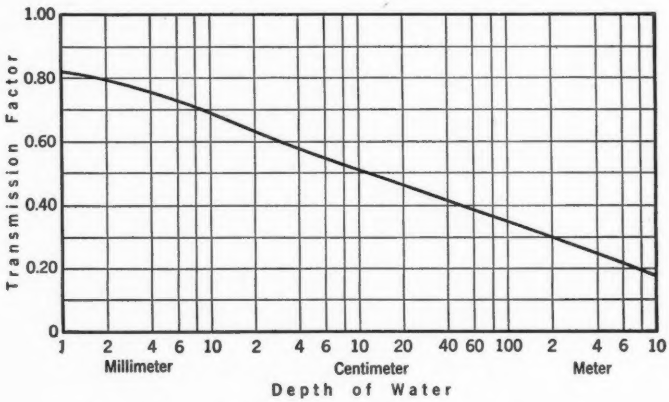

FIG. 12.—TRANSMISSIVITY OF WATER FOR SOLAR RADIATION

will give good results in laboratory work if properly shielded. It is not convenient for most installations since suitable potentiometers and recording equipment are expensive. Measurement of temperature with a copper-constantan thermocouple to the nearest tenth degree Fahrenheit requires measurement of potential with an accuracy of 2 microvolts. Satisfactory correlation of field observations probably can be obtained by the use of a thermometer with a cylindrical bulb placed horizontally in the upper quarter inch of water. It may be necessary to support it by means of a float where evaporation exceeds 0.1 in. per day to insure that the bulb is always submerged. Of course, the instrument should be shielded.

The most consistent correlation of evaporation rate with vapor-concentration difference has been found in the work of those experimenters who used pans heated from beneath rather than pans heated at the surface by solar radiation. Undoubtedly this is due largely to the fact that the experimenters on outdoor pans used average daily temperature, but may also be due, in part at least, to the difficulty of measuring the actual surface temperature.

Measurement of air temperature and relative humidity offers no particular difficulty. Suitable recording instruments are standard meteorological equipment. Care should be taken, however, that the measurements are made at a sufficient distance from the pan to be representative of conditions at a distance.

It is desirable to measure the wind velocity close to the surface of the pan, since practically all field installations as well as experimental laboratory tests have been made this way. The standard Weather Bureau anemometer as installed at Class A evaporation pans should be satisfactory.

The daily fluctuations of barometric pressure have a very slight effect on the evaporation rate. Barometric pressure needs to be considered only when

comparing results obtained from installations at different altitudes. For this purpose, the average pressure corresponding to the altitude is sufficiently accurate.

Type of Evaporation Pan.—The pan itself has considerable effect on the evaporation and it should be standardized more carefully than has been common in past practice. It should be as large as can be handled conveniently since the change of evaporation rate with change of diameter becomes less as the diameter increases, making the results more reliable. The evaporation from a large pan also approaches more closely the evaporation from a lake, although this is not necessarily an advantage.

Mr. Powell's experiments show that the ratio of rim height to diameter has considerable effect on the evaporation rate. Fig. 11 indicates that a change in rim height for a given diameter has least effect on the evaporation rate when the ratio of rim height to diameter is about 0.1. Greater evaporation with an increased height of rim has been noted but has been variously attributed to the greater amount of solar radiation intercepted, and to evaporation from the water film drawn up the side of the pan by capillarity. The effect of rim height for a floating pan may be different from that for an exposed land pan. Further experimentation on this subject appears to be required. Until better information is available, it would seem desirable to maintain the rim height, or the depth of water in the pan, as nearly constant as possible.

The color and surface characteristics of the pan walls have an effect on the evaporation rate because of the variation in the absorptivity. The effect is not constant, but varies seasonally, although over a year's time a black pan will evaporate more water than a white one. For comparable results, pans should have similar surfaces which should be carefully maintained. The natural oxidation of a galvanized surface changes its absorptivity. Failure to clean pans frequently results in the growth of algae and a change in the characteristics of the surface.

Considerable amounts of heat are transferred through the sides and bottom of an uninsulated evaporation pan. It is evident that the ratio of wall area to surface area will affect the evaporation rate. Similar pans exposed under identical atmospheric conditions, but at different latitudes, will show differences in evaporation rate caused by the change in the angle of incidence of solar radiation on water surface and pan walls. The differences found between floating and buried pans as compared with an exposed pan can be accounted for by the difference in the amount of heat transferred through the pan walls. The amount of heat transferred can be calculated with approximate accuracy for the exposed and floating pans where the air and water temperatures, respectively, are known. It is much more difficult for the buried pan where the thermal conductivity of the soil is uncertain, varying with the moisture content, and the soil temperature is constantly fluctuating. The evaporation from a uninsulated pan is also affected by changes in the absorptivity of the outer wall of the pan. These difficulties can be most easily avoided by the use of an insulated pan. Further, the insulated pan has characteristics similar in many ways to a large body of water. Since heat transfer through the walls and bottom is minimized, the boundary or edge effect from this cause is made

very small and the evaporation from the pan area approaches that from the larger body. The corrections depend largely on the pan geometry rather than on the meteorological conditions, which vary both locally and seasonally.

There seems to be no argument in favor of an uninsulated pan except cheapness. The cost of insulating a pan is not great. It may be argued that records from the standard Class A pan have the advantage of being comparable with each other. In view of the manner in which the data accompanying evaporation are obtained and recorded, it is doubtful if comparisons of results from Class A pans have much value.

In this connection, the writer wishes to call attention to an admirable discussion by Charles H. Lee, M. Am. Soc. C. E. (42), of the most important factors affecting evaporation, the shortcomings of the Class A pan, and the general confusion attending the subject of evaporation. Although first published in August, 1926, it appears not to have attracted the attention it deserves. Two quotations may not be out of place:

> "The writer's experience has led to the conclusion that in order to use data obtained from the Class A land pan it is necessary to determine a convection factor, which will differ for every locality or meteorological condition."
>
> "* * * much of the confusion and uncertainty which exists regarding evaporation rates is due to a failure to consider the fundamental factors in choosing observational methods."

Suggestions for Further Study

It is suggested that further study of the subject of evaporation from the standpoint of practical use be along the following lines:

1. The establishment of a standard evaporation station with accompanying measuring instruments designed to recognize the fundamental principle of evaporation and to measure the significant variable factors in such a way that they may be useful in correlating the observational results.

2. A study of the probable heat-transfer losses through the walls of uninsulated pans for an average day for each month of the year at all stations where there are significant changes in air temperature, humidity, or insolation. Such a study would make it possible to recompute the recorded evaporation in terms of that from an insulated pan. Although the original data are not in the best form, it is possible that a much better correlation of existing data could be obtained in this manner than is now available.

3. A study of the relationships between the existing data and those obtained from the new standard pan with the purpose of correlating all records as far as possible.

Acknowledgment

This paper is a condensation and revision of a dissertation by the writer entitled "Evaporation from a Limited Free Water Surface." It was submitted in July, 1939, in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Mechanical Engineering in the Graduate Division of the University of California.

The writer wishes particularly to acknowledge the interest, encouragement, and advice offered during the study by Professors L. M. K. Boelter, M. P. O'Brien, M. Am. Soc. C. E., and L. B. Loeb of the University of California, and the cooperation of the Tennessee Valley Authority through A. S. Fry, M. Am. Soc. C. E., Head Hydraulic Engineer, in the use of its experimental evaporation apparatus.

A copy of the dissertation, containing observational data, details of calculation, and additional technical discussions, has been placed on file in the Engineering Societies Library[2a] for further reference.

APPENDIX I

Bibliography on Evaporation

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APPENDIX II

Nomenclature

In dealing with material drawn from many sources, the meanings of symbols inevitably become confused necessitating changes in the notation used by the original authors. The following symbols, defined where they first appear, conform essentially with American Standard Letter Symbols for Heat and Thermodynamics Including Heat Flow,[3] prepared by a Committee of the American Standards Association, with Society participation, and approved by the Association in 1943:

A = area of water surface;
a = spacing of etched reference lines on a glass plate (see also b);
B = barometric pressure;
b = a constant expressing the form of a wind correction factor; as $a + b\,V$;
C = coefficient in Eqs. 10 and 11;
C_R = coefficient;
c = unit vapor concentration; Δc = concentration difference;
c_p = specific heat at constant pressure;
D_v = diffusivity;

[3] ASA – Z10.4 – 1943.

E = depth of evaporation;
e = base of natural logarithms;
F = a wind correction factor (Eq. 18);
G = a barometric pressure correction factor (Eq. 19);
g = acceleration of gravity;
H = elevation above ground surface (Eq. 16);
h = coefficient of heat transfer;
h' = coefficient of mass (vapor) transfer;
K = coefficient in Eqs. 7, 13, and 14;
K' = coefficient in Eq. 6*c*;
K_o = von Kármán's universal roughness coefficient;
k = thermal conductivity;
L = characteristic dimension of a surface; diameter of an evaporation pan; length of a cylinder;
l = length of lever arm of interferometer evaporimeter;
N = number of interference bands; also a dimensionless number (Eqs. 6); when primed, these dimensionless numbers characterize mass transfer;
ΔN = change in number of interference bands (Eq. 1);
N_{Nu} = Nusselt's number;
N_{Gr} = Grashof's number;
N_{Re} = Reynolds' number;
N_{St} = Stanton's number;
n = exponent in Eqs. 6 and 7;
p = unit pressure;
p_S = pressure of saturated vapor at the temperature of the water surface;
p_H = pressure of saturated vapor at the temperature of the dewpoint;
q = rate of heat transmission;
r = radius;
s = height of a collar on a cylinder;
T = absolute temperature;
t = thermometric temperature;
V = wind velocity;
w = flow rate of vapor;
w_A = flow rate of vapor per unit area;
x = length along path of heat flow or vapor transfer;
Z = coefficient of roughness in Eq. 16;
β = coefficient of expansion;
γ = unit weight of mixture of air and water vapor; $\Delta\gamma$ = difference in unit weights;
λ = wave length of light;
μ = coefficient of absolute viscosity;
ρ = mass density.

Dimensionless Moduli.—Dimensionless groups have been used freely throughout the paper. Because of their usefulness in classifying physical phenomena and their independence of dimensions, they are widely used for

correlation of experimental data. In applying these dimensionless groups, physical properties were evaluated at the arithmetic mean temperature of the fluid.

Grashof's number—

$$N_{Gr} = \frac{L^3 \, \Delta t \, \beta \, \gamma^2}{g \, \mu^3} \qquad (21a)$$

—characterizes the movement of a gas or liquid under the influence of inertia forces and the buoyancy caused by temperature differences. It is associated with heat transfer by convection. Grashof primed—

$$(N_{Gr})' = \frac{L^3 \, \gamma \, \Delta\gamma}{g \, \mu^3} \qquad (21b)$$

—is similar to the Grashof number but characterizes convection associated with mass transfer.

Nusselt's number—

$$N_{Nu} = \frac{h \, L}{k} \qquad (22a)$$

—characterizes the phenomenon of heat conduction between a solid and a gas or liquid. Nusselt primed—

$$(N_{Nu})' = \frac{h' \, L}{k} \qquad (22b)$$

—is similar to Nusselt's number but characterizes mass transfer between a solid and a gas, as in evaporation.

Reynolds' number—

$$N_{Re} = \frac{V \, \rho \, L}{\mu} \qquad (23)$$

—characterizes the movement of a gas or liquid under the influence of inertia and viscous forces.

Stanton's number—

$$N_{St} = \frac{k}{\mu \, c_p \, g} \qquad (24a)$$

—characterizes the conduction of heat into a viscous fluid which is in motion. Stanton primed—

$$(N_{St})' = \frac{D_v \, \gamma}{\mu \, g} \qquad (24b)$$

—is similar to Stanton's number but is applicable to mass transfer into a moving viscous fluid, as evaporation into air.

AMERICAN SOCIETY OF CIVIL ENGINEERS

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DISCUSSIONS

PRINCIPLES OF DEPRECIATION

REPORT OF THE SPECIAL COMMITTEE AUTHORIZED BY THE BOARD OF DIRECTION TO ANALYZE AND DISCUSS THE 1943 REPORT OF THE NATIONAL ASSOCIATION OF RAILROAD AND UTILITIES COMMISSIONERS' COMMITTEE ON DEPRECIATION

Discussion

By Donald Gunn

Donald Gunn,[27] Assoc. M. Am. Soc. C. E.[27a]—The report of the Special Committee of the Society brings to the depreciation controversy a greatly needed new approach to the problem. The suggestion that actual depreciation, measured or appraised from time to time, be made the bench mark for treatment of this problem is, at least, in accord with its true nature. This approach certainly offers the best prospects for determining depreciation in a manner consistent with the equities of all parties at interest.

During recent years, the NARUC committee and many others have devoted much effort to numerous methods based upon functions of age and life. Actuarial mathematics has been brought to bear on the problem. These efforts and statistical refinements would be valuable if the problem were of a statistical nature, but since it is not, the value of the effort is meager. It can be truly said that, if there were a statistical method, a statistical solution could be had —if there were some statistics.

The writer proposes to discuss neither the mathematics nor the detailed conclusions of the NARUC report. It is his conviction that the basic fallacy of the conclusions of that report lies not in the details but rather in the fundamental assumptions on which the structure of details is erected.

Note.—This report was published in June, 1944, *Proceedings*. Discussion on this report has appeared in *Proceedings*, as follows: September, 1944, by C. Beverley Benson, E. M. T. Ryder, Wallace B. Carr, J. L. Campbell, Paul L. Holland, J. Kappeyne, C. Terry duRell, Anson Marston, Terrell Bartlett, E. E. Hart, and Luther R. Nash.

[27] Chf. Engr., Pennsylvania Water & Power Co., Baltimore, Md.

[27a] Received by the Secretary August 24, 1944.

Stripped of all theoretical refinements and hypothetical assumptions, the conclusions of the NARUC report relating to the use of the age-life theory of depreciation (the term, "age-life theory of depreciation," is intended to include both the method of calculation and the so-called consistent treatment of the reserve balance as recommended by the NARUC report) rest upon two primary basic assumptions. These assumptions are:

(*a*) That money is invested in utility enterprises on the proposition of receiving simply "a return on and return of the funds"; and

(*b*) That actual depreciation at a given time cannot be determined with as much accuracy as remaining life can be assumed.

Numerous other assumptions are subsidiary to these and are frequently at odds with each other. The writer believes each of the primary assumptions to be basically unsound and that the assumptions, as a whole, fail as support for the NARUC report conclusions.

The first proposition of "a return on and a return of funds" sounds innocent enough as a generalization. It seems to be a fair statement of the basis for borrowing and lending money on security. It is perhaps a proper basis for self-liquidating government enterprises or private enterprises based upon resources subject to depletion. However, it certainly is not the basis on which normal equity investments are made. This feature, itself, would seem to be the primary point of difference between governmental and private-enterprise investments.

Entirely aside from theoretical considerations, no "return of funds" actually takes place, as assumed or asserted by the NARUC report conclusions. The corporate charters of most—if not all—public utilities are perpetual and make no provision for the repayment of investments. Dividends may not be declared out of funds collected as depreciation. Thus, the "return of funds" as a fact can be accomplished only through a liquidating dividend upon dissolution of the enterprise. That the retirement of debt does not constitute a "return of funds" has been clearly shown by Samuel Ferguson.[28]

In spite of its obvious fallacy, this theory of investment is, in fact, the sole basis for rationalizing all age-life reasoning with respect to depreciation. Without this single assumption, the inescapable circle of age-life reasoning is broken. Then all theories of depreciation, based merely upon the passage of time, lose their apparent logic. Consequently, the use of depreciation, derived from age-life theories as a part of the rate-making formulas, becomes illogical.

On the contrary, if the theory of "a return on and a return of invested funds" were the general basis for private-enterprise investments, then each and every assumption as to the occurrence of depreciation with respect to the passage of time can be rationalized. It can readily be shown, mathematically, that every such assumption as to a relationship between the occurrence of depreciation and the passage of time fully satisfies that economic theory of investment on the often-repeated points of equity and consistency. Not only is this true of the methods covered by the NARUC report but it will be true

[28] "Depreciation Accounting as Applied to Public Utilities," by Samuel Ferguson, The Case, Lockwood & Brainard Co., Hartford, Conn., 1943.

of all possible future theories which relate the occurrence of depreciation merely to the passage of time.

Therefore, if the proposition of "a return on and a return of invested funds" prevails, there is small reason to argue that one age-life depreciation theory which satisfies that proposition is right and that another age-life theory which satisfies it with equal mathematical precision is wrong. A selection among the various age-life methods, therefore, must be based upon grounds other than the theoretical logic inherent in the method. The writer believes that any and every theory of depreciation will be vitally affected by the economic theory on which investments are made. It is suggested, therefore, that the proposition that investments in private enterprise are made simply for "a return on and a return of the invested funds" should be examined with particular care.

Now, as to the second basis, the assumption "that actual depreciation at a given time cannot be determined with as much accuracy as remaining life can be assumed" was not stated in these words by the NARUC committee. However, this assertion as to the accuracy and reliability of estimates of remaining life is usually made and frequently accepted as an article of faith by age-life advocates. The writer is convinced that the NARUC committee reached that conclusion because, otherwise, some of the most important conclusions of its report would be wholly without logic. This conclusion of the NARUC committee suggests that it is possible to guess or assume the time and scope of future events with greater accuracy and more reliability than the impact of past events can be measured or evaluated. It would seem to the writer that the mere statement of that proposition would have disposed of the matter but the conclusions of the NARUC report prove the contrary.

With minor and obscure qualifications, the NARUC report recommends the straight-line age-life theory of depreciation for all purposes. It even proposes to use that theory, applied retrospectively, as the test of adequacy of present reserves and as a basis for future rates. These latter two recommendations are altogether baseless unless they are founded upon the assertion that remaining life can be assumed with greater accuracy than actual depreciation can be determined at a given time. Only if that proposition is accepted can those recommendations be rationalized, even as an abstract theory.

The advocates of this straight-line theory first undertook to argue that depreciation actually proceeded uniformly with respect to time. They finally became convinced that probably as much as 80% of the past retirements of public utility property resulted from so-called functional causes aside from wear and tear. A large part of this 80% was actually attributable to obsolescence. Since nonphysical depreciation obviously does not progress uniformly with respect to time, that line of argument has been largely discarded. The argument, itself, became afflicted with obsolescence.

A later argument was based upon equity. Its advocates assert, in effect, that the depreciation reserve really represents reimbursement by the ratepayer for property used up in rendering service. This reserve may be temporarily invested in property additions or in certain securities but it may not be paid out in dividends. Nevertheless, it is urged that the full rate of return on the

utility's property with respect to the total reserve should be credited to the ratepayer's bill. This is the effect of the straight-line theory of depreciation.

The NARUC committee and other extreme advocates of the straight-line theory of depreciation for rate-making purposes have gone one step further. They do not seriously concern themselves with whether or not the reserve was created from past charges against operating expenses. They urge not that the ratepayer receive a fair return credit on the amount of the actual reserve but rather that he receive it on what would have been the amount of the reserve if what they currently consider a "reasonable life table" had been followed from the beginning on a straight-line basis. This retrospective depreciation theory is not even consistent with the equity and consistency argument ordinarily advanced in support of the straight-line method. If this theory has a rational foundation, it must be in the sanctity of assumed future lives.

The straight-line theory, whether or not retroactively applied, imputes to life estimates a degree of reliability that no estimate involving the future can ever possess. This undue emphasis on life tables and the tendency to treat such estimates as facts prompt an inquiry into their essential character. Such estimates are made up of two components—namely, a determinable present age and an indeterminable future life. The future life, in the case of most equipment, is and must remain a pure guess. The potential accuracy of this guess for most equipment is greatly influenced by two factors:

(1) The distance the estimate extends into the future; and
(2) The degree to which obsolescence enters into the estimate.

The first factor is perhaps more a matter of philosophy than of fact. It is, however, none the less real and important. Perhaps its attendant uncertainties are about the same as those of philosophy. The second factor presents the greater problem. When the forces which bring about obsolescence are considered, it is obvious that estimates of future life involve the piling of one assumption upon another without let or hindrance. It is well to remember in this connection that there are no metes and bounds to the field of assumptions as to the future life of most utility property.

The future occurrence of obsolescence results from the future trend of a multitude of forces. The most important of these which are commonly encountered are:

(*a*) Efficiency of equipment that may be invented and manufactured in the future;
(*b*) Operating cost of equipment that may be invented and manufactured in the future; and
(*c*) Installed cost of equipment that may be invented and manufactured in the future.

Estimates of the rate at which obsolescence will proceed are based almost wholly upon these factors. Having made assumptions as to these, the length of time until property becomes obsolete—that is, has reached the end of life—is a matter of how much obsolescence has already taken place. It is, there-

fore, obvious that even the formulation of a rational estimate of life presupposes the ability to determine the actual depreciation at a given time. Stated more directly, the proposition becomes: Actual depreciation plus the effect of assumed future probabilities. This is the only rational basis of a life estimate for most utility property.

Sensing the weakness of unsupported assumptions of future life, an effort has been made by advocates of age-life theories of depreciation to relate these assumptions to statistics. It is a fact that there are few statistics of sufficiently wide applicability. Also, it is true that if there were enough such statistics, the use of them in connection with the straight-line method would entail further violent and unsound assumptions. In this connection, it is frequently asserted that statistics on equipment can be relied upon the same as statistics on human life. The basic fallacy of this proposition is quite obvious. Human beings remain substantially unchanged from generation to generation; whereas, in a developing industry, one generation of equipment bears little, if any, similarity to a succeeding generation. One company's experience would be no more applicable to another company than the statistics of one race to another.

In spite of all these, the NARUC committee has, in effect, concluded that remaining life can be determined with such accuracy and reliability that it should be the sole determinant of depreciation, past as well as future. It would seem that the simple statement of that proposition would convince any one of the fallacy in the straight-line theory. For more than thirty years, the courts have recognized this fallacy and have repeatedly expressed a preference for estimates of actual depreciation.

The writer believes that the extent of depreciation at a given time can be evaluated with more accuracy and greater reliability than future life can be estimated or guessed. It seems, therefore, that such evaluations, and not estimates of future life, should govern the treatment of this problem. Such evaluations govern the decisions to create and destroy property. Those decisions are irrevocable. It seems altogether natural and logical that such evaluations should govern the treatment of depreciation during the life of the property, for they are only intermediate stages between these final decisions.

The factors that cause depreciation are well known to engineers, for they are constantly applied by them in selections and replacements of facilities. Measurements of actual depreciation, like the selection and replacement of property, are essentially comparative procedures in which an item of existing property is compared in a proper manner with an available modern item of property. In such a comparison, the various points of difference, such as efficiency, reliability, etc., are evaluated. Technical data and experience are available that can be utilized in making such valuations. There are also numerous technical and scientific data, tests, and procedures that are valuable aids in the determination of physical condition.

The justification for creating a depreciation reserve in a continuing enterprise is the maintenance of the integrity of the investment. This has been done when a reserve balance equal to the actual depreciation is maintained. As a matter of security, a reasonable margin of safety or contingency above

that amount should be maintained in the reserve. It is suggested that this reserve balance be governed by determinations of actual depreciation made at appropriate intervals of perhaps five years. The amount charged annually against operating expenses could be readily fixed and adjusted according to a graduated scale, predetermined to restore variations in the reserve balance at any desired rate.

There are only two really troublesome aspects of the problem:

(1) How much shall be collected each year through rates, and
(2) How shall the balance in the reserve be treated in fixing the rate base?

In the writer's opinion, the importance of the specific accuracy of the annual amount has been stressed out of all proportion to its real significance. It can be adjusted from time to time to meet the circumstances. The annual amount is merely a means to an end; the end sought is the creation and maintenance of an adequate reserve. It is this end result which is of primary importance. The means by which it is brought about is comparatively unimportant to either the investor or the ratepayer, as long as it is reasonable and orderly.

The treatment of the reserve balance in connection with the rate base has long been the most controversial aspect of depreciation. Any balance in the depreciation reserve to the extent it has been created by charges against operating expenses after a fair return has been earned should be taken into account in fixing rates. It is important, however, that this be done in a manner which is economically sound and just to all parties at interest. In the writer's opinion, the NARUC committee's proposal to deduct whatever balance may be found in the reserve or, perhaps, what they may now think should have been there, possesses none of these attributes. There is no real justification for such a procedure, whether the basis of rates be derived from reproduction cost, book cost, original cost, or any other basis.

Treatment of the depreciation reserve as a negative component has been advocated on two theories, namely:

(*a*) That the reserve balance represents a reimbursement of capital; and
(*b*) That the ratepayer is entitled to the rate of return credit for the earnings on the reserve.

The fallacy of the first reason has been thoroughly exposed by Mr. Ferguson.[28] The fallacy of the second reason for considering the reserve balance a negative component has not been so completely dealt with. However, it can be very simply demonstrated.

Treating the reserve balance as a negative component of the rate base is simply a device for crediting the ratepayer with hypothetical earnings equal to the rate of return. Even though the funds represented by the reserve were always completely and immediately invested in plant, they would not be entitled to a credit of the full rate of return for several reasons. First, the rate of return is still supposed to contain an increment above the cost of money as a reward for good management. Certainly, the ratepayer is not the manager and, therefore, is not entitled to that increment, however much or little it may be. There is another and much more substantial reason why the rate-

payer is not entitled to a credit of the full rate of return. This relates to the fact that reserve funds invested in property do not take the full risk of the enterprise. Certainly, dividends on stock cannot be paid until provision for depreciation has been made. It follows, therefore, that the depreciation funds are entitled to a smaller return.

Whether these funds are entitled to a higher or lower rate than the bonded debt is a difficult question. The writer knowns of no clear-cut theory upon which this question can be resolved. There is, however, a very persuasive practical consideration which inclines the writer to the opinion that the proper rate of earnings should not exceed that of the bonded debt. In the event of financial disaster, stockholders will lose their investment entirely, and these depreciation funds invested in the property will be dissipated last in rendering service beyond the time when earnings meet actual expenses.

By far the greater part of the controversy over depreciation flows from its treatment in connection with the basis of earnings. Most of this controversy could be resolved by crediting the ratepayer with reasonable earnings on the reserve balance. At least, the area of reasonable disagreement as to reasonable earnings of the reserve is much smaller than in the case of estimates of future life. There are also more adequate standards of measure for earnings based upon given risks. The writer is convinced that the treatment of this perplexing problem could be much simplified by removing it from any relationship to the rate base by crediting the ratepayer directly with reasonable earnings on any balance in the depreciation reserve which has been created by charges against cost of service in excess of a fair return.

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DISCUSSIONS

FLOOD FORMULAS BASED ON DRAINAGE BASIN CHARACTERISTICS

Discussion

By Howard M. Turner, and Byron O. McCoy

Howard M. Turner,[38] M. Am. Soc. C. E.[38a]—The title of this interesting paper does not indicate the complete subject covered, because it omits the treatment of flood frequency which is a very interesting part. The authors have succeeded in deriving a set of formulas which take into account more characteristics of the drainage area than any others for river floods, and which show very good correlation with similar floods computed by unit hydrographs. These formulas should prove of great value for general use in regions similar to Massachusetts. Many problems arise, particularly on small drainage areas where there are not sufficient records for drawing unit hydrographs or where the problem does not justify a complete analysis. In such cases, formulas like these which take into account the factors that affect the shape of the unit hydrograph will be of great use.

The authors give the qualifications as to the use of these formulas in certain specific cases, emphasizing the fact that when there is a large storage the flood should be computed and then routed through the storage. It would be interesting to know whether they have tried to apply these formulas to any other rivers than those given in the paper and whether they consider the formulas limited to streams of less than 500 sq miles. It would seem probable that they may be generally applied to New England and similar regions, as well as Massachusetts.

These formulas cover four flood magnitudes described as minor, major, rare, and maximum and further defined as of 15-yr, 100-yr, 1,000-yr frequency and a maximum possible flood. Methods of computing flood frequency are open to great question due to the difficulty of extending the frequencies considered far beyond the normal records of the past. It is true that referring to

Note.—This paper by H. B. Kinnison, M. Am. Soc. C. E., and B. R. Colby, Esq., was published in March, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: June, 1944, by Clarence S. Jarvis, and September, 1944, by L. R. Beard, and M. J. Ord.

[38] Cons. Engr., Boston, Mass.; Lecturer Water Power Eng., Harvard Univ., Cambridge, Mass.

[38a] Received by the Secretary September 5, 1944.

floods as minor or major, rare or maximum, brings in a frequency element, but this is entirely different from assigning a definite period of years to a given sized flood. The authors recognize the unsatisfactory results of the usual methods of computation and the wide variation in results obtainable. They introduce a method of estimating frequency on a somewhat different basis from the usual figures, which are based on past records of peak flow extended by plotting or by some mathematical process.

The major differences in the authors' methods are the use of precipitation records and flood runoff figures instead of stream-flow records of peak floods to obtain frequency curves. This is, in effect, an adaptation of the rational method used for small areas. The writer believes that this method is a definite advance in the studies of flood frequency and should give more consistent results. There is, however, the same difficulty in the extension of records to periods many times in excess of their actual length and the handling of the records of major storms occurring in a short period.

There are also certain objections to the authors' method. The use of flood runoff rather than peak discharge has certain disadvantages—chiefly the question of what is to be included in the runoff for different floods. An example of this is the 1936 flood in New England where there were two peaks due to two storms, in part superimposed and combined with a large amount of melting snow. In theory, the runoff should be a better value by which to express the magnitude of floods as it would make floods on rivers with different characteristics much more comparable. By means of the unit hydrograph, flood runoff can be converted satisfactorily to peak flow, but this presupposes a normal flood due to a single storm without large base-flow complication often affected by runoff from melting snow.

It seems to the writer, also, that the use of precipitation records for frequency computation of flood flows is open to question. It is true that precipitation records are longer and thus serve better to establish frequency relations; but the other factors bring in another unknown which may more than offset the advantage of the long record—that is, the further question of the condition of the ground during any given storm. For example, there is the case of the 1936 flood cited, where the ground was frozen and the runoff from the rainfall was combined with a large runoff from melting snow. There was the 1927 flood in which two weeks of previous rainfall had filled up the level of the water in the ground; and on the other hand there is the large storm of September, 1932, in New England, which produced no serious floods, as it came after a long period without rain. The necessity of allowing the additional factor to cover the question of how much of a given storm runs off introduces an uncertainty which, it seems to the writer, more than makes up for the advantage of using the longer storm records instead of peak flood flow records.

In their use of the unit hydrograph to convert from the flood runoff to peak flow, the authors point out an interesting relation, indicating that the distribution graph should show a larger peak for large floods than for small floods. It would be advisable to investigate this further. The authors have found it so from the rivers they studied but it should be noted that these are all small rivers—with one exception less than 500 sq miles. It would be interesting to

know whether they have made any study to see whether the same relation holds for larger rivers, and also whether this tendency is not greater on the smaller rivers and tends to increase as the size of the drainage area increases. The distribution graphs given in U. S. Geological Survey *Water-Supply Paper No. 772* [39] do not show this; but these are mostly for larger areas.

The authors' frequency data lead them to classify floods with definite figures of 15, 100, and 1,000 years for the minor, major, and rare floods. This frequency of the 15-yr flood can be determined definitely, and that for the 100-yr flood, fairly satisfactorily in many cases. There is no difficulty in the use of some maximum figures, provided the maximum is defined as in this paper. There is clearly an important flood value between the "major" floods of 100-yr frequency and the maximum flood. The authors have used a "1,000-yr flood" computed by their method.

It would be better to leave out the definite figure of 1,000 years and merely call the flood "rare." Frequency computations may help in obtaining comparable floods on the different rivers for these formulas as the authors have done, but other methods for obtaining comparable floods on different streams could be used. One would be on the basis of flood runoff—a rare flood could be expressed as one which produces the equivalent of a definite flood runoff of, say, 8 in. on 500 sq miles, with corresponding adjustment for larger or smaller areas; or the rare flood could be expressed in a percentage increase over the 100-yr flood, without using the definite 1,000-yr figure.

The difficulty with the 1,000-yr frequency, expressed as such, is that there are so many ways of computing frequencies. There is the authors', as given in this paper, besides numerous others, all of which give widely different results, depending not only on the method of estimating but also on the length or the period of the record, which is much more difficult. Thus, when the term "1,000-yr flood" is used, it really should be combined with the statement of how it is computed. For example, one should say, to be definite, the 1,000-yr flood according to Fuller, based on a 25-yr period, 1918–1943; or the 1,000-yr flood according to Hazen, based on a 100-yr record, 1843–1943.

Something of the difficulties involved in finding these frequency relations is shown in the paper itself. The curves in Fig. 12 show frequencies for the 1938 flood as follows: Millers River near Winchendon, 680 years; Middle Branch of Westfield River at Goss Heights, 1,000 years; and Housatonic River near Great Barrington, 25 years. This appears to be too wide a difference, even when the smaller rainfall on the latter area is considered.

Any one who has been through the work necessary to analyze the floods on any stream will realize the work that has gone into this paper. The authors are to be congratulated for their success in expressing the effect of the various characteristics of the drainage area on the flood peak in a formula of usable form. This should prove a valuable check on the other methods particularly where more or less special characteristics are involved.

[39] "Studies of Relations of Rainfall and Run-Off in the United States," by W. G. Hoyt, *Water-Supply Paper No. 772*, U. S. Geological Survey, 1936.

BYRON O. MCCOY,[40] ASSOC. M. AM. SOC. C. E.[40a]—The formulas presented by the authors will be of much assistance to engineers in and around Massachusetts. A tremendous amount of time and effort has been expended on the method presented in this paper, and the engineering profession is fortunate in having this additional tool for its use. It would be desirable to make similar investigations in other states. Further correlation with other watershed characteristics may be disclosed in regions with soil, vegetal cover, temperature, and other conditions differing from those found in Massachusetts. As with all such empirical equations, it is well to observe the authors' caution that the formulas should not be applied where conditions are materially different from those on which the formulas are based.

The exponents applied to the various physical characteristics used in the formulas vary somewhat with frequency. To be entirely consistent, for a given watershed, the variation of magnitude with frequency should be a function of probabilities. It does not seem reasonable that the effect of drainage area or distance the water must travel should be less on the major flood than on the minor or rare floods.

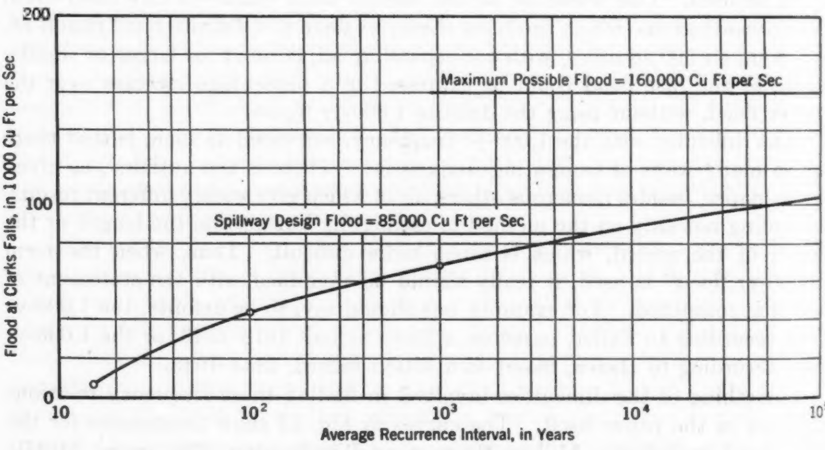

FIG. 18.—PROBABILITY OF OCCURRENCE OF CLARKS FALLS SPILLWAY DESIGN FLOOD BASED ON THE KINNISON AND COLBY FORMULAS

The data on maximum possible floods are of much interest. As defined by the authors, the maximum flood is one (see heading, "Maximum Flood Runoff") "* * * that could not be exceeded under any possible * * * combination of circumstances." For all practical purposes it has a probability of occurrence of one only. As the authors state, a design based on (see heading, "Conclusions") "* * * less than a maximum flood should be expected to involve some risk * * *."

As an indication of the risk being assumed in present practice, Fig. 18 shows the probability of occurrence of the design flood for the Clarks Falls dam re-

[40] With Charles T. Main, Inc., Boston, Mass.

[40a] Received by the Secretary September 14, 1944.

cently built on the Lamoille River in Vermont. The spillway drains an area, M, of 701 sq miles, with a median altitude above the dam, s, of 880 ft, and an average distance the runoff must travel, L, of 43 miles. It was assumed that the authors' formulas would apply, and the 15-yr, 100-yr, and 1,000-yr floods have been plotted in Fig. 18. A curve drawn through these three points and extended to become asymptotic to the maximum flood of 160,000 cu ft per sec at infinity indicates that the spillway design flood of 85,000 cu ft per sec has a probability of occurrence of about once in seven thousand years.

The authors' method of deriving magnitude-frequency relations should prove very valuable. Heretofore, probability studies based on past records for a particular stream have been inadequate because they did not take into account the probability of occurrence on that stream of the requisite meteorological and hydrological conditions to produce a given flood.

Magnitude-frequency relations are often used by the engineer to evaluate economic benefits of flood control projects; but, beyond a period of fifty years or so, the annual benefit becomes so small that additional expenditure for relief from such rare floods is usually difficult to justify on an economic basis. For floods of greater magnitude, the engineer becomes interested in the probability of occurrence—that is, the chances of occurrence, rather than frequency.

The authors have assumed that major floods and larger floods in Massachusetts will be caused by a 72-hr storm and that the chances of such a storm occurring over any drainage area in Massachusetts are one in one. They have also assumed that the chances are one in one that the distribution of runoff generation will follow the pattern they have used. Then, from the frequency, or probability, of a given rainfall and the probability of that rainfall producing a given runoff, the authors have obtained the probability of the resulting flood. The method is an important contribution in arriving at the true probability of occurrence of a given flood at a given station.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

ECONOMICAL CANAL CROSS SECTIONS

Discussion

BY JULIAN HINDS

JULIAN HINDS,[6] M. AM. SOC. C. E.[6a]—The author has introduced a new member into the family of criteria for "best shapes" of water conduits. Engineers generally are familiar with "best shapes" as determined from minimum area and minimum hydraulic radius alone, and know that only for pipe lines and flumes do these shapes have a great deal of practical significance. For lined excavated conduits (the type considered by the author), the best hydraulic shape usually is poor from a cost-of-construction viewpoint. The engineer is interested in the cheapest and most practical conduit rather than the one with the smallest waterway area and perimeter. For this reason, constructed canal sections usually bear little resemblance to the theoretically "best sections."

If all of the elements affecting costs were included in the theory, a complete formula for the "cheapest section" might be written. Generally, the results are so involved and conditions so variable from place to place that it is preferable to depend on trial-and-error procedure. However, knowledge of the theory of best shape is useful.

Modern development of machines for placing concrete lining has stabilized, or at least limited, some of the variables. The author has taken advantage of this fact to move a step nearer complete determination of the best shape for this particular type of conduit. This is a laudable proposal and the results deserve the careful consideration of engineers interested in problems of this kind.

Although the author's statement of basic principles is clear, the writer wishes to restate them from a slightly different point of view. Consider a differential mass of concrete standing alone on a sloping bank, as at dl, Fig. 1. This mass tends to move down the slope under the influence of gravity. Its movement is resisted by friction along the surface of the bank. If the sliding force exceeds the resisting force, the mass will move unless some other form of support is provided.

NOTE.—This paper by Victor L. Streeter, Assoc. M. Am. Soc. C. E., was published in May, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: September, 1944, by V. A. Endersby.

[6] Gen. Mgr. and Chf. Engr., The Met. Water Dist. of Southern California, Los Angeles, Calif.

[6a] Received by the Secretary August 28, 1944.

If the slab is filled in above and below the mass in question, as indicated in Fig. 1, the differential mass may be made stable by the compressive strength of the concrete below it. The top surface of the mass must in turn support the accumulated excess of sliding force over resistance for the concrete above. Thus the required crushing strength of the differential block will increase as its position moves downward from the top. If the slope is straight, the maximum is reached at the bottom or wherever an outside support occurs. If the slab is curved, the maximum occurs at the position where the individual mass is just stable within itself (against sliding), that is, where $\theta = \theta_0$, using the author's notation.

If the unit weight of the concrete, the resistance to sliding, and the form and dimensions of the bank are known, the required internal resistance can be computed; or, if the permissible internal stress is known, the allowable slope and dimensions may be determined.

The author finds the total force to be resisted in a unit width of slab down to $\theta = \theta_0$ by integration. This he divides by the thickness, t, to get unit stress, and by the weight of a cubic unit of the concrete to get a "steepness factor." The last division is not essential. An "allowable stress" is perhaps more comprehensible to most engineers than a "steepness factor."

It is noted that Eq. 2*a* gives σ in linear units—that is, in feet, for American usage. If the weight is left in, the results are in pounds per square foot, which might be converted to pounds per square inch. The numerical values of σ from Eq. 2*a* are very nearly equal to the allowable stress in pounds per square inch if the canal dimensions and t are in feet, as a 1-in. by 1-in. by 12-in. prism of concrete weighs approximately 1 lb.

The writer is of the opinion that making the sliding resistance, F, constant, regardless of slope, is incorrect. In other sliding problems, the resistance is assumed to be proportional to the normal force against the supporting surface. If this change is accepted, F becomes $F' \cos \theta$ and Eq. 2*a* becomes

$$\sigma' = y_0 - x_0 \tan \theta_0 \qquad (26)$$

in which x_0 is the horizontal projection of the portion of the lining above the point where $\theta = \theta_0$.

If applied to a vertical concrete wall, Eq. 26 gives a stress at the bottom of the wall equivalent to that corresponding to the weight of the wall. For Eq. 2*b*, the stress (or steepness factor) is less than half that corresponding to the weight of the wall; that is, more than half the weight of the lining would have to be held by friction on the vertical bank. The author perhaps does not contemplate a vertical wall but it comes within the theoretical scope of his equations. Also, the effect noted applies in lesser degree to other slopes. Maximum slopes for some of the channels of Table 2 appreciably exceed 1 : 1. The resistance offered by securely anchored, imbedded steel may be more or less independent of slope, which might call for a compromise between Eqs. 2*a* and 26 for heavily reinforced linings. The writer knows of no data on which such a compromise could be based.

Changing from Eq. 2*a* to Eq. 26, or to a compromise, if found desirable, would not vitiate the general idea of the paper but would change some of the detailed computations.

Apparently a placing machine with a tailpiece or apron to smooth and temporarily support the freshly placed lining is contemplated. It is also presumably contemplated that concrete has some resistance to deformation when first placed, and that this resistance increases with passing minutes. Therefore, the slower the machine moves, other things being equal, the stronger will be the concrete as it clears the apron. Thus the speed of machine operation, the height and slope of the bank, and the strength of the soft concrete are interrelated.

Ignoring for the moment the probability that Table 2 should be revised to conform to Eq. 26, it is possible to select the "best" of the listed sections in any of the groups, Table 2(*a*), 2(*b*), or 2(*c*). However, no means for choosing between the three groups is offered. Such a choice requires a knowledge of the details of the machine, the original consistency and rate of hardening of the concrete, and the effect of speed on the cost of lining. The details of the machine are known or determinable. The consistency of the concrete and the rate of stiffening, for the first 10 min or so, are controllable to some extent and are no doubt ascertainable under fixed conditions. The effect of permissible speed of placement on the cost of lining requires careful study. The costs of materials, of trimming banks, of troweling surfaces, and of other items are more or less independent of speed. Nevertheless, other cost items are reduced by rapid placement. By balancing, properly, all the cost and time factors for lining against variations in excavation quantities and other items, if there are any, a "cheapest" section can be devised. Here, as in the more usual case of "best shape," topography and other variables are likely to complicate theoretical formulas beyond the limit of practicability, requiring resort to trial-and-error methods.

As a matter of fact, the average computer will find most of the equations beyond Eq. 8 too complicated for routine use. This is not because the theory is complicated but because of involved mensuration. The use of these equations in a theoretical study, such as that presented in this paper, is entirely permissible, but "practice" is likely to revert to "trial and error."

Another practical point that should be considered is that a good lining machine is costly and durable. Unless the job to be done is large, it will be desirable to salvage the machine for subsequent use or to use a machine salvaged from previous work. This can be accomplished more readily with a trapezoidal machine than with one of circular, parabolic, catenary, or other curved form. Curved fillets of unchanging radius at the bases of the trapezoidal side slopes are permissible and beneficial.

Everything considered, the trapezoidal section—possibly with slight modification—is likely to remain the preferred section for excavated concrete-lined canals. However, it is important that "everything" actually be "considered," in order that the extent of the sacrifice to practicability may be evaluated fully. To this end this paper will be a helpful one.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

WEDGE-SHAPED STRUCTURAL MEMBERS UNDER DIRECT STRESS AND BENDING

Discussion

BY PHIL M. FERGUSON, AND STANLEY U. BENSCOTER

PHIL M. FERGUSON,[7] M. AM. SOC. C. E.[7a]—Sometimes the reasonableness of a conclusion is as reliable a measure of an equation's worth as the reasonableness of the assumptions underlying this equation. As one who accepts conclusions very reluctantly when they do not appear to lead to reasonable results, the writer confesses he had difficulty with this paper. The following comments are largely critical of the results obtained by the proposed method.

The author has attacked a complex problem in an interesting manner. However, when the reader reaches Eq. 14, defining the location of the neutral axis, and discovers that this equation involves the slope of the reinforcing steel but not the slope of the compression face of the concrete, questions are raised. According to this equation, no matter what the concrete face slope (angle ϕ in Fig. 1) may be, the neutral axis will be the same relative distance $k\,d$ from the compression face; and this equation then leads to the further conclusion that f_0 and f_s are also independent of ϕ. The writer does not believe that this is true. On the contrary, he believes that all three of these terms are influenced by ϕ, f_0 and k being influenced to a very significant degree. It is probable that the influence of either ϕ or θ, which defines the slope of the steel, will be small when these angles are small; but it also seems probable that the influence is considerable when either angle is large. It is chiefly for these larger angles that concrete designers need specialized formulas.

It is probable that the basic assumption that plane sections before bending remain plane after bending is not acceptable in this problem, as satisfactory as it may have proved in other cases involving members of constant depth. It may be that this is the source of the following serious discrepancies.

Another contributing factor may be an inconsistency in the treatment of deformations at the beginning of the paper. The author is careful to relate

NOTE.—This paper by Robert B. B. Moorman, Assoc. M. Am. Soc. C. E., was published in June 1944, *Proceedings*.

[7] Chairman, Civ. Eng., Univ. of Texas, Austin, Tex.

[7a] Received by the Secretary August 21, 1944.

the deformation of the steel to a certain vertical deformation and to his angle α. He makes no corresponding attempt to relate the outside concrete fiber deformation to the angle α. Instead, Eq. 3 states a supposed relation between f_0 and α which assumes that the unit concrete deformation vertically will be given by f_0/E_c. This relation ignores the fact that, with the principal stress parallel to the sloping concrete face, there exists a compressive stress perpendicular to f_0; and this stress will be large for large values of ϕ. It would seem that Poisson's ratio can scarcely be excluded from this equation. Even if this modification were made, it would seem safer to establish α or $k\,d$ from the inclined edge deformation which is more surely known in terms of f_c. A different location for the neutral axis might result from this treatment. However, it is doubtful whether any treatment based on straight-line deformations can be used in the general fashion indicated by this paper when face angles are of interesting magnitude.

The author implies that no limitations on the relative or absolute values of ϕ and θ are necessary. In the author's illustrative problem shown in Fig. 7, horizontal section A–A might have been taken at some other slope provided proper adjustment in *e*, *N*, *V*, and *p* accompanied this different choice of section. If the theory presented in this paper is correct, the concrete stress f_c at the left face at point A should be equally well determined by the author's section or by other inclined sections through the same point. Since the author's

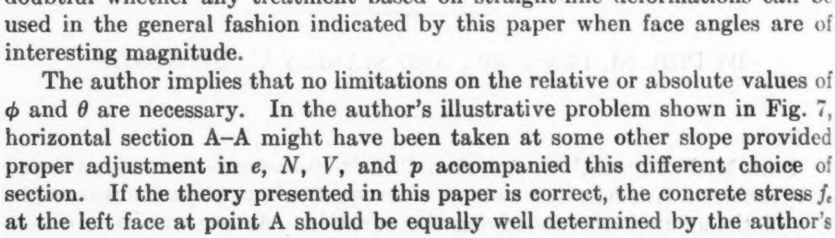

FIG. 8

conclusions appeared open to considerable doubt, the writer calculated f_c at point A by the methods of this paper using sections A–B and A–C (Fig. 8(*a*)). The author reported f_c to be 639 lb per sq in. when based on section A–A. The calculated value of f_c at the same point on section A–B is 810 lb per sq in., and on section A–C it is 595 lb per sq in. If this theory were correct, there would not be this spread of more than one third in the resulting value of f_c when only the slope of the section analyzed is altered. One must conclude

that the formulas are not correct for all sections—possibly not correct for any section.

To investigate this matter further, another wedge-shaped member was loaded in simple fashion, as shown in Fig. 8(*b*). With the load acting along one face, comparative analyses were simplified. The maximum concrete stress f_c at point A was calculated by the author's method applied to sections A–A′, A–B, and A–C. The steel stress f_s at point B was also calculated with sections A–B and D–B. The depth of section A–B was designated as d_1; the area of steel A_s was taken as $0.01\ b\ d_1$. The results show even more spread than in the foregoing case. A correct theory should give identical values of f_c and identical values of f_s for the points shown in Table 1.

TABLE 1.—Discrepancies in Calculated Stresses

Based on Section: Fig. 8(*b*)	Description	Computed f_0	Computed f_c at point A	Computed f_s at point B
A–B	Perpendicular to steel	$2.05\ P/b\ d_1$	$8.20\ P/b\ d_1$	$12.86\ P/b\ d_1$
A–A′	Horizontal	$3.22\ P/b\ d_1$	$4.32\ P/b\ d_1$	....
A–C	Perpendicular to compressive face	$2.67\ P/b\ d_1$	$2.67\ P/b\ d_1$	....
D–B	Perpendicular to compressive face	....	....	$51.6\ P/b\ d_1$

Such divergent results lead to the conclusion that, until this method can be modified to give more consistent results on different sections, or until those sections are designated on which it may properly be used, the method must be rejected as one producing inconsistent results.

No lengthy comment will be made on the bond and shear formulas since these formulas must be subject to the errors of the bending formula. However, attention should be directed briefly to the theory behind the shear formula, because it might lead to unsafe design even when based on a correct tension formula. The author uses the shear adjacent to the steel as a measure of diagonal tension. This shear is not the largest shear in the case of a beam subjected to direct stress and bending. A larger shear exists within the compressive area, even in the case of a parallel sided member.[8] Moreover (although this is contrary to the conclusion of Stanley U. Benscoter and Samuel T. Logan, Juniors, Am. Soc. C. E.[8]), this larger shear will produce a considerably larger diagonal tension in the compression zone near the neutral axis than exists between the neutral axis and the steel. The small compression stress acting in this zone does make the diagonal tension smaller than the shear, but still leaves it greater than the lower shear value found near the steel. The shear adjacent to the steel is thus not a proper measure of the maximum diagonal tension and the author should not have used it as such.

S. Timoshenko[9] shows that large shears will accompany bending of a wedge-shaped beam of homogeneous material. In the case of a 15° slope of each face he shows that the maximum unit shear on a section perpendicular to the axis

[8] "Shear and Bond Stresses in Reinforced Concrete," by Stanley U. Benscoter and Samuel T. Logan, *Proceedings*, Am. Soc. C. E., March, 1944, p. 295.

[9] "Strength of Materials," by S. Timoshenko, Pt. II, 1941, p. 337.

of symmetry is some 90% greater than on a beam of the same depth with parallel faces. This result does not necessarily apply to a reinforced concrete beam; but it should make one very conservative in the calculation of unit shears to be used as a measure of diagonal tension in the case of any member with sloping sides.

STANLEY U. BENSCOTER,[10] JUN. AM. SOC. C. E.[10a]—The concrete designer will find in this paper a group of formulas, for wedge-shaped members, developed on the basis of reasonable assumptions. Due to the lack of such formulas, internal stresses have been commonly computed for wedge-shaped members by using formulas and methods which should be applied only to prismatic members. The writer has been guilty of such erroneous stress analysis and has witnessed the same error by many designers. The Joint Committee Code renders little assistance in the analysis of combined stresses in wedge-shaped members, and it is of real value to the profession that formulas should become available.

It is possible to study the individual terms of some of the formulas and equations developed in this paper and thereby to establish some rules for design which tie in closely with past experience in the design of prismatic members. The cubic equation which governs the location of the neutral axis for a prismatic beam is well known. An examination of Eq. 14 indicates that one may replace the tensile steel area A_s, or the tensile steel area ratio p, by $A_s \cos \theta$ or $p \cos \theta$, respectively, and the cubic equation for prismatic beams will become the cubic equation for a tapered beam. This gives a simple rule for design. Project the normal cross section of the steel rods into the plane of the beam cross section to obtain a reduced steel area. Then calculate the neutral axis location just as for a prismatic beam. This rule is as applicable to compressive steel as to tensile steel. In the case of compressive steel, $(A_s)'$ and p' are replaced by $(A_s)' \cos \phi$ and $p' \cos \phi$, in which $(A_s)'$ and p' refer to the compressive steel area.

In solving the cubic equation for k a designer usually estimates the first significant figure correctly but seldom guesses the second significant figure correctly in an initial estimate. The graphs of Fig. 4 will give the second significant figure for members without compressive steel and will probably give a good estimate in most cases for members with compressive steel. They are equally useful for prismatic or tapered members. The third significant figure may be obtained from a few cycles of trial and error or by iteration computations. In research studies of shear and bond stresses the normal stresses must be computed accurately at sections 1 in. apart and a fourth significant figure for k is generally desirable.

Before considering the individual terms of the formulas for shear and bond stresses it is profitable to recall the formulas recommended by the Joint Committee Code for wedge-shaped members without thrust. The usual formulas for shear and bond stress for a prismatic beam may be used if the shear V is

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[10a] Received by the Secretary August 31, 1944.

replaced by the modified shear V_1 given by,[11]

$$V_1 = V \pm \frac{M}{d} (\tan \theta + \tan \phi) \qquad (34)$$

in which the negative sign is to be used when the depth of the beam increases in the direction of increasing bending moment; and θ and ϕ are the angles, respectively, that beam faces make with a direction normal to the direction of the external shear. It is not difficult to see that this definition of θ and ϕ is sensible only in the case of a beam without thrust and cannot be extended to a beam-column such as a member of a rigid frame. In the case of a haunched beam-column, loaded normal to its straight surface, it is common practice to calculate stresses on a cross section normal to the loaded surface, thus assuming θ to be zero in regions of negative moment. Perhaps it would be more accurate to calculate the stresses on a plane that is normal to the bisector of the wedge angle. No matter what direction of cross section is considered by the designer as the one on which flexural theory will give the most accurate stress determination, the resultant force may be resolved into a normal and tangential component. The expression "external shear" creeps into the Joint Committee Code at several points, thus causing considerable confusion.

If the coefficient $\frac{M}{d}$ in Eq. 34 is replaced by $\frac{M}{j\,d}$, the formula has a simple rational explanation. This explanation has been given by the Joint Committee Code by stating that[12] "the resultant internal shear at any section is increased or decreased by the vertical components of the inclined stresses (tension or compression) * * *." Since $\frac{M}{j\,d}$ equals the normal tension T' or compression C' on a section without thrust, Eq. 34 may be modified to read (using the negative sign):

$$V_1 = V - (T' \tan \theta + C' \tan \phi) \qquad (35)$$

The normal forces T' and C' are components of the inclined forces T and C. Thus the terms $T' \tan \theta$ and $C' \tan \phi$ are components of T and C in the plane of the cross section. The force T acts along the center line of the tensile bar (or bars) whereas the force C acts parallel to the external surface in the compressive region. It would appear more reasonable if the angle ϕ were replaced by the angle $-\delta$ as introduced by Mr. Moorman, so that C acts along the line of centroids of the compressive stresses. The formula for δ as given by Eq. 33 may also be used for members without thrust.

It may be shown that Eq. 29*a* for v can be expressed in terms of the modified shear V_1 as given by Eq. 35. Regrouping the terms of Eq. 29*a* gives,

$$v = \frac{V}{b\,j\,d} - \left[\frac{N\,(e - j\,d)}{b\,(j\,d)^2} \tan \theta + \frac{N\,e}{b\,(j\,d)^2} \tan (-\delta) \right] = \frac{V_1}{b\,j\,d} \qquad (36)$$

in which

$$V_1 = V - [T' \tan \theta + C' \tan (-\delta)] \qquad (37)$$

[11] "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete," Rept. of the Joint Committee on Standard Specifications for Concrete and Reinforced Concrete, *Proceedings*, Am. Soc. C. E., June, 1940, Pt. 2, p. 56.

[12] *Ibid.*, p. 55.

When a section carries thrust as well as bending, the forces T' and C' are not equal but must be computed separately. Eq. 37 gives the designer a convenient rule for determining the shear and bond stresses in agreement with the assumptions of Mr. Moorman's analysis.

There are upper limits on the angles θ and ϕ beyond which the assumption of linear strain provides too much error. It is also known from the theory of elasticity that, in wedge-shaped beams, the shearing stresses have distributions which are greatly different from those in prismatic beams. The error involved in neglecting the rate of change of k along the member in evaluating δ from Eq. 32 is not known. These matters remain for future research study. It may be noted that, according to Eq. 33, the angle δ becomes zero when θ and ϕ are zero. Actually the angle δ is not zero for a prismatic beam that carries thrust. This error enters in due to the neglect of the rate of change of k along the member in writing Eq. 32. Neglecting the rate of change of j along the member leads to the Joint Committee Code formula, Eq. 34, and the formula is thus slightly in error.

Corrections for *Transactions:* In June, 1944, *Proceedings*, page 885, line 3, change "ϕ" to "$-\phi$"; and, on page 886, line 6, change "$6{,}390 \frac{0.71}{0.29} = 15{,}630$" to "$5{,}960 \frac{0.71}{0.29} = 14{,}590$."

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DISCUSSIONS

SHEAR EFFECT ON THE STRENGTH OF STRUTS

Discussion

BY S. SERGEV

S. SERGEV,[18] ESQ.[18a]—An error in analysis was made in overlooking the discontinuity in slope of the strut axis caused by shearing forces. (The check method used was not independent as subsequent examination revealed.) Discontinuity of slope due to shearing action in members of homogeneous material may result from discontinuities of shearing forces and cross-sectional areas, separately or in combination. In the strut under discussion, both factors enter into consideration because the shearing forces and cross-sectional areas above and below the point of loading are different. For the special application, when the strut cross section is constant throughout, there is a sudden change in the shearing forces at the point of loading, and consequently a sudden change in the shearing strains is to be expected. Thus, the final slopes due to bending and shearing actions above and below the point of loading are different from each other and a kink in the elastic curve exists.

To introduce the correct concept of slope relationships at points of discontinuity into the analysis, again consider Eq. 20. The constants C and D may be evaluated from the conditions (compare Eq. 21):

$$(y_a)_{x_a=0} = 0; \quad \text{and} \quad (y_a)_{x_a=a} = e \qquad (63)$$

from which $D = 0$, and $C = \frac{e}{a} + \frac{P\,e\,a^2}{6\,E\,I_a\,L}$. Substitution of these values into Eq. 20 gives (compare Eq. 23):

$$y_a = -\frac{m^2\,e\,x^3_a}{6\,u\,L} + \left(\frac{e}{a} + \frac{m^2\,e\,a^2}{6\,u\,L}\right) x_a \qquad (64)$$

NOTE.—This paper by S. Sergev was published in November, 1943, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1944, by Leon Beskin, and Marshall Holt; and March, 1944, by David B. Hall.

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[18a] Received by the Secretary July 17, 1944.

This expression for y_a will be simpler to use in the strain energy which follows later.

The interrelationship between the slopes of the upper and lower segments can be established using the following arguments: Consider a small part of the strut in the vicinity of the loading (Fig. 10) under the action of the shearing forces as indicated. The line mn represents the tangent to the elastic curve if the shearing strains are neglected. The angles between the line mn and the vertical are numerically equal but opposite in sign, the latter fact following from the choice of the coordinate axes. This is the requirement of continuity for bending actions only. However, if it is desired to consider shear deformations, then the shearing strains in the upper and in the lower segments must be taken into account, and the following equality, with due regard to signs, should be used:

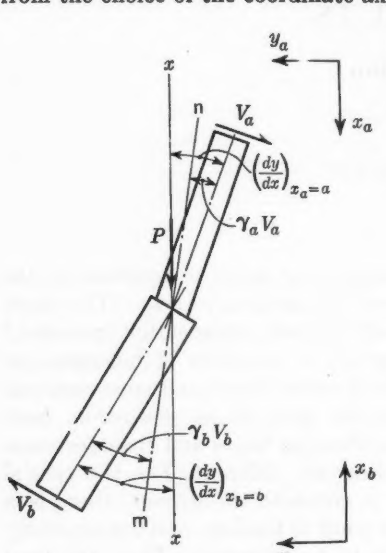

FIG. 10

$$\left(\frac{dy_b}{dx_b} + \gamma_b V_b\right)_{x_b=b} = -\left(\frac{dy_a}{dx_a} - \gamma_a V_a\right)_{x_a=a} \dots (65)$$

In Eq. 65:

$$\gamma_b V_b = \frac{K V_b}{A_b G} = \frac{K}{A_b G}\left(-P\frac{dy}{dx} + \frac{P e}{L}\right)_{x_b=b} \dots (66a)$$

and

$$\gamma_a V_a = \frac{K V_a}{A_a G} = \frac{K}{A_a G}\frac{P e}{L} \dots (66b)$$

Substituting these equivalents into Eq. 65, keeping in mind that

$$\left(\frac{dy}{dx}\right)_{x_b=b} = \frac{e\,a\,q\cot q\,b}{L} + \frac{e}{L} \dots\dots\dots\dots\dots (67a)$$

and

$$\left(\frac{dy}{dx}\right)_{x_a=a} = -\frac{m^2 e\,a^2}{2\,u\,L} + \frac{e}{a} + \frac{m^2 e\,a^2}{6\,u\,L} \dots\dots\dots\dots (67b)$$

gives

$$\frac{e\,a\,q\cot q\,b}{L} + \frac{e}{L} + \frac{K P}{A_b G}\left(-\frac{e\,a\,q\cot q\,b}{L} - \frac{e}{L} + \frac{e}{L}\right) = -\left(-\frac{m^2 e\,a^2}{2\,u\,L} + \frac{e}{a} + \frac{m^2 e\,a^2}{6\,u\,L} - \frac{K P e}{A_a G L}\right) \dots\dots\dots\dots (68)$$

Eliminating e and noting that $\frac{K P}{A_a G} = \frac{A_b}{A_a}\left(1 - \frac{m^2}{q^2}\right)$, Eq. 68 may be written

as follows:

$$a\,q \cot q\,b = \frac{q^2\,a^2}{3\,u} - \frac{q^2}{m^2}\left(1 + \frac{L}{a} - \frac{A_b}{A_a}\right) - \frac{A_b}{A_a} \quad\text{.........}(69a)$$

Let $b = k\,L$, $a = L\,(1 - k)$, and, rearranging slightly, Eq. 69a becomes

$$L\,q \cot k\,q\,L = \frac{q^2\,L^2\,(1 - k)}{3\,u}$$

$$- \frac{q^2}{m^2}\left[\frac{2 - k}{(1 - k)^2} - \frac{A_b}{A\,(1 - k)}\right] - \frac{A_b}{A_a\,(1 - k)} \quad\text{.........}(69b)$$

which should replace Eq. 24b as first derived. It can now be seen that not only does the ratio of the moments of inertia u enter into consideration but also the ratios of the cross-sectional areas $\frac{A_b}{A_a}$. Thus, it is possible to study a strut with $I_a = I_b$ and with the areas A_b and A_a unequal.

The result given in Eq. 69b completely describes the buckling criteria for struts of uniform but unequal cross sections in each segment loaded as previously described (see Fig. 1).

Check by Strain Energy.—If the equations for the elastic curve of the strut axis are established in both segments, then a solution of the problem may be obtained without resorting to the relationship expressed in Eq. 65, by using the strain energy method. If it is admitted that under a critical load P a slight deflection of the strut is possible, then a small amount of strain energy of flexure and shear will be stored in the strut in the form of potential energy ΔV. This potential energy must be equal numerically to the loss in potential energy of the force P due to loss in elevation.

The potential energy of strain can be calculated in the usual way from the following expression:

$$\Delta V = \int_0^b \left(\frac{M^2_b}{2\,E\,I_b} + \frac{K\,V^2_b}{2\,A_b\,G}\right) dx_b + \int_0^a \left(\frac{M^2_a}{2\,E\,I_a} + \frac{K\,V^2_a}{2\,A_a\,G}\right) dx_a \quad\text{..}(70)$$

in which

$$M_b = -\,P_y + \frac{P\,e}{L}\,x = -\,\frac{P\,a\,e \sin q\,x}{L \sin q\,b} \quad\text{............}(71a)$$

$$V_b = -\,P\,\frac{dy_b}{dx_b} + \frac{P\,e}{L} = -\,\frac{P\,a\,e\,q \cos q\,x}{L \sin q\,b} \quad\text{...........}(71b)$$

$$M_a = -\,\frac{P\,e\,x}{L} \quad\text{.......................}(71c)$$

and

$$V_a = -\,\frac{P\,e}{L} \quad\text{.........................}(71d)$$

Substituting these equivalents into Eq. 70, and performing the indicated operations, give

$$\Delta V = \frac{m^4\,E\,I_b\,e^2}{L^2}\left(\frac{a^2\,b}{4 \sin^2 q\,b} + \frac{a^2 \cot q\,b}{4\,q} + \frac{a^3}{6\,u} + \frac{K\,a\,E\,I_b}{2\,A_a\,G}\right.$$

$$\left. + \frac{a^2\,q^2\,b\,K\,E\,I_b}{4\,A_b\,G \sin^2 q\,b} + \frac{K\,a^2\,q^2\,E\,I_b \cot q\,b}{4\,q\,A_b\,G}\right) \quad\text{.............}(72)$$

The loss in potential energy of the force P is equal to the product of the force and the distance that it moves downward. This loss in potential energy will be designated by ΔU, and it can be calculated in the following manner:

$$\Delta U = \int_0^b P\,(ds_b - dx_b) = P \int_0^b \left[\sqrt{1 + \left(\frac{dy_b}{dx_b}\right)^2} - 1 \right] dx_b \ldots. (73)$$

If higher powers than the first of the small quantities are neglected, the loss in potential energy becomes

$$\Delta U = P \int_0^b \frac{1}{2}\left(\frac{dy_b}{dx_b}\right)^2 = \frac{P}{2}\int_0^b \left(\frac{a\,e\,q\cos q\,x}{L\sin q\,b} + \frac{e}{L}\right)^2 dx_b \ldots.. (74a)$$

or

$$\Delta U = \frac{m^2\,E\,I_b\,e^2}{L^2}\left(\frac{a^2\,q^2\,b}{4\sin^2 q\,b} + \frac{a^2\,q^2\cot q\,b}{4\,q} + a + \frac{b}{2}\right) \ldots\ldots (74b)$$

Setting Eqs. 72 and 74*b* equal to each other and simplifying, an expression identical to Eq. 69*b* is obtained.

This check method is not entirely independent, as it uses the results obtained from the geometrical properties of the elastic curve of the strut. Its merit lies in the fact that it is independent of the discontinuities of slope at the point of loading.

Approximate Solution.—A check of the results previously obtained can be made by using an approximate method based on strain energy and an assumed shape for the elastic curve. (A check of this kind may not reveal errors in analysis if the influence of the errors on the solution is small, say, less than 1%.) As a first approximation, assume that the deflection curve is a sine curve,

$$y = f_1 \sin\frac{\pi\,x}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (75)$$

in which f_1 is a constant. The general expression for the strain energy is that expressed by Eq. 70, in which the bending moments and shearing forces will be now expressed as follows:

$$M_b = -P\,y + \frac{P\,e\,x_b}{L} = -P\,f_1\sin\frac{\pi\,x}{L} + \frac{P\,e\,x_b}{L} \ldots\ldots. (76a)$$

$$V_b = \frac{P\,e}{L} - P\,\frac{dy_b}{dx_b} = \frac{P\,e}{L} - \frac{P\,f_1}{L}\cos\frac{\pi\,x_b}{L} \ldots\ldots\ldots.. (76b)$$

$$M_a = -\frac{P\,e\,x_a}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (76c)$$

$$V_a = -\frac{P\,e}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (76d)$$

and

$$e = f_1 \sin\frac{\pi\,b}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (76e)$$

Substituting Eqs. 76 into Eq. 70, and performing the indicated operations,

the expression for strain energy becomes

$$\Delta V = \frac{P^2}{2\,E\,I_b}\left[\frac{b\,f^2_1}{2} - \frac{f^2_1\,L}{4}\sin\frac{2\,b\,\pi}{L} + \frac{e^2\,b^3}{3\,L^2} - \frac{2\,f_1\,e\,L}{\pi^2}\sin\frac{\pi\,b}{L}\right.$$

$$\left. + \frac{2\,f_1\,e\,b}{\pi}\cos\frac{\pi\,b}{L} + \frac{e^2\,a^3}{3\,u\,L^2}\right] + \frac{K\,P^2}{2\,A_b\,G}\left[\frac{b\,e^2}{L^2} + \frac{f^2_1\,\pi^2}{L^2}\left(\frac{b}{2} + \frac{1}{4}\sin\frac{2\,b\,\pi}{L}\right)\right.$$

$$\left. - \frac{2\,f_1\,e}{L}\sin\frac{b\,\pi}{L} + \frac{e^2\,a\,A_b}{L^2\,A_a}\right] \qquad (77)$$

The loss in potential energy ΔU of the load P due to a lowering of its point of application can be calculated as previously described. The expression for ΔU is

$$\Delta U = P\int_0^b \frac{1}{2}\left(\frac{dy_b}{dx_b}\right)^2 dx_b = \frac{P\,f^2_1\,\pi^2}{2\,L^2}\int_0^b \cos^2\frac{\pi\,x_b}{L}\,dx_b$$

$$= \frac{P\,f^2_1}{2\,L}\left(\frac{\pi\,b}{2\,L} + \sin\frac{2\,\pi\,b}{L}\right) \qquad (78)$$

Substituting $e = f_1 \sin\left(\frac{\pi\,b}{L}\right)$ into Eq. 77, and equating it to Eq. 78, a general expression for P may be obtained. If in this expression $b = \frac{L}{2}$ is used, it will give, after a slight rearrangement,

$$P = \frac{\pi^2\,E\,I_b}{L^2}\,\frac{1}{\left(\frac{7}{6} - \frac{8}{\pi^2} + \frac{1}{6\,u}\right) + \frac{K\,E\,I_b}{A_b\,G\,L^2}\left(\pi^2 - 6 + \frac{2\,A_b}{A_a}\right)} \qquad (79)$$

in which, for struts with solid webs, $\frac{K\,E\,I_b}{A_b\,G\,L^2}$ is denoted by α and for latticed struts by β.

Special Application.—As a special application of the formulas previously derived, consider the case in which $I_b = I_a = I$, $A_a = A_b = A$, and $k = \frac{1}{2}$. Substitution of the values into Eq. 69*b* gives

$$6\,L\,q\cot\frac{1}{2}\,q\,L = q^2\,L^2 - 24\,\frac{q^2}{m^2} - 12 \qquad (80)$$

Let $m\,L = t$ and $q\,L = w$; then Eq. 80 may be transformed into Eq. 46, in which for the latticed struts $\beta = \frac{1}{t^2} - \frac{1}{w^2}$. The roots of Eq. 46 were determined by Mr. Beskin.

Now consider Eq. 79. Making the substitutions noted herein and simplifying gives the following relationship between P and β,

$$P = \frac{\pi^2\,E\,I}{L^2}\,\frac{1.913}{1 + 11.23\,\beta} = \frac{1.012\,P'}{1 + 11.23\,\beta} \qquad (81a)$$

As a function of ψ, the critical load,

$$P = \frac{\pi^2 E I}{L^2} \frac{1.913}{1 + P' (0.602 \psi)} = \frac{1.012 P'}{1 + 0.602 \psi P'} \qquad (81b)$$

in which P' is the critical load on the strut similarly loaded but neglecting shear deflections; that is,

$$P' = \frac{(4.32)^2 E A}{\left(\frac{L}{r}\right)^2} \qquad (82)$$

The degree of accuracy of Eq. 81*a* will be studied now by comparing the critical load P obtained from it with that obtained from Eq. 46. Using the data in the numerical example following Eq. 35*b*, and the $\frac{L}{r}$ ratios in Table 1, the critical loads determined by Eqs. 46 and 81*a* may be ascertained (see Table 2). From these calculations it can be concluded that, for practical applications, Eq. 81*a* gives sufficiently accurate results.

TABLE 2.—NUMERICAL EXAMPLE; SPECIAL APPLICATION

$\frac{L}{r}$ (1)	β (2)	t (3)	P (4)	P_1 (5)	P' (6)	$\frac{P}{P'}$ (7)	$\frac{P_1}{P'}$ (8)
60	0.0145	3.99	0.0905 E	0.0922 E	0.106 E	0.855	0.87
80	0.00815	4.13	0.0546 E	0.055 E	0.0597 E	0.915	0.92
100	0.00522	4.2	0.0361 E	0.0366 E	0.0382 E	0.945	0.96
120	0.00362	4.24	0.0255 E	0.0258 E	0.0265 E	0.962	0.973
160	0.00204	4.28	0.01465 E	0.01475 E	0.0149 E	0.983	0.99
200	0.00131	4.295	0.00942 E	0.00951 E	0.00955 E	0.986	0.995
Source	Eq. 37*b*	Fig. 5	Eq. 34*a*	Eq. 81*a*	Eq. 34*b*	Exact	Approximate

An advantage of the approximate method is the fact that it gives explicit functions such as Eq. 81*a* or 81*b* for evaluating the critical load. The more exact method reduces to an implicit function, Eq. 46, the roots of which must be found first before it can be used practically.

In Col. 7, Table 2, ratios of the buckling load on latticed struts to that on struts with solid webs are given using the more exact analysis. Similar ratios are given in Col. 8, Table 2, where the buckling load on the latticed struts was determined by the approximate method. When the results in Cols. 7 and 8 are compared, the approximate method slightly underestimates the shear effect on the strength of the strut but is sufficiently accurate for most practical purposes. Finally, comparing the final results in Table 1, Col. 7, with the corrected values as shown in Table 2, Col. 7, the conclusion is that the solution, as first given, exaggerated the shear effect on the strength of the struts.

Comment on Discussions.—Mr. Beskin's criticism that the extended treatment of the conditions involved in Assumption 5 is superfluous is partly justified. Eqs. 1 to 3 were intended only to show the approximate relationship existing between curvature and the corresponding bending moment. It would have been logical to have included the shear effect on the final curvature.

To make Eqs. 8 and 17 general a constant should be added to the right side. This constant is found from the boundary conditions, is dependent on both the shear and moment effects, and in the general case can be determined only after the final solution is obtained. It is the angle through which the member must be rotated so that the shear deflection will satisfy the end condition. Since the shearing force V is affected by this rotation, Eq. 8, as altered, may be considered fundamental but not always useful as a starting point in the solution of a given problem. In straight simple beams of constant cross section symmetrically loaded, the constant is zero. However, if the loading is symmetrical but the cross section varies, it is necessary to rotate the member about one end so that the shear deflection will vanish at the other. A simple beam of constant cross section unsymmetrically loaded by a concentrated force also requires that the constant be zero as do simple beams of constant cross section loaded by couples, the latter showing distortion in the longitudinal direction.

The separation of the deflection y into components is permissible provided those components are properly evaluated. In problems to which the principle of superposition does not apply, these components cannot be found independently and resort is made to adding together the elastic properties of the member which are free from constants not known in advance. For the problem at hand, the addition of the curvatures due to the corresponding shear and moments is such a sum.

The expression $\frac{K\,V}{A\,G}$ represents the relative sliding angle of two adjacent transverse cross sections of the member and is the average effect of the shearing stresses over the entire cross section. The constant K is found by the energy method equating the work of the shearing force to the total work of the shearing stresses, or by using the more refined methods of the theory of elasticity.[19]

The mode of restraint at the ends of a member plays a significant rôle in the solution of shear deflection problems, and can only be treated adequately by the more exact methods developed in the theory of elasticity. Experience shows, however, that comparatively simple problems in elasticity become extremely difficult or defy solution altogether except by approximate methods. For practical needs, the approximate methods often give the desired accuracy.

Mr. Beskin is correct in calling attention to the misstatement that "the curvature due to bending of a built-up strut is independent of the type of lattice used." If (as implied from the outset) there are many lattice bars in the length of the member, the average curvature may seem continuous and independent of the type of lattice. Actually it is not.

The author laid no claim on originality in handling the problem as the references show. The prime purpose was to ascertain quantitatively the influence of the shearing forces on the strength of the type of struts under consideration.

Mr. Holt calls attention to the important practical problem of estimating the distance between the principal parts of the built-up strut for equal strengths

[19] "Theory of Elasticity," by S. Timoshenko, 1st Ed., McGraw-Hill Book Co., Inc., New York and London, 1934, pp. 42 and 150.

about the principal axes, and develops a relationship between the radii of gyration. Regretfully, because of the error already noted, Fig. 6 may need to be modified.

Mr. Hall early found the error in analysis and much later submitted his discussion with convincing evidence that a change in the affected equations is warranted. His treatment of the comparison of the corrected values with those derived by the writer is interesting and conclusive.

In conclusion, the writer feels that, although the continuity of the paper has been disturbed by the error in analysis, this defect is compensated for by again emphasizing the discontinuity of slopes in elastic bodies due to shearing forces. The writer appreciates the participation of the discussers. At this time it also seems appropriate to thank D. H. Young for his interest. Modesty prevented his sending a discussion because the writer called attention to the error in advance.

Corrections for *Transactions:* In November, 1943, *Proceedings*, on page 1453, at the end of the line following Eq. 20, insert footnote 4*a*: "The second condition in Eq. 21 has been found incorrect; see discussions and Eq. 65"; after "values" in the line following Eq. 22, insert footnote 4*b*: "See Eq. 64"; on page 1454, at the end of the line preceding Eq. 24*a*, insert footnote 4*c*: "See Eq. 69*a*"; at the end of the line following Eq. 24*a*, insert footnote 4*d*: "See Eq. 69*b*"; in the ordinate values of Fig. 3 change "$t = \frac{m}{L}$" to "$t = m\,L$"; invert Fig. 4; on page 1459, line 3, after "are given," insert footnote 6*a*: "See Table 2"; in February, 1944, *Proceedings*, on page 225, add P to the last term in Eq. 40*b*; and, in March, 1944, *Proceedings*, on page 416, line 11, change "force $P\,e$," to "force $\frac{P\,e}{L}$." See also errata in February, 1944, *Proceedings*, on page 230.